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Grade 6 Math Circles

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Units and Dimensional Analysis

Introduction

__________________ is the removal of a common factor in the numerator (top) and denominator (bottom) of a fraction. Here is an example of __________________:

$$(2)\left(\frac{8}{10}\right) = \frac{(2)(8)}{10} = \frac{(2)(8)}{(2)(5)} = \frac{\cancel{(2)}(8)}{\cancel{(2)}(5)} = \frac{8}{5}$$

We can also think of __________________ as recognizing when an operation is undone by an _______________ operation. The most common example we will see are the opposite operations of multiplication and division. In the example above, the division by 2 is undone by the multiplication by 2, so we can cancel the factors of 2. If we skip writing out all the steps, we could have shown the cancelling above as follows:

$$(2)\left(\frac{8}{10}\right) = \cancel{(2)}\left(\frac{8}{\cancel{(2)}(5)}\right) = \frac{8}{5}$$

We can also cancel variables in the same way we cancel numbers. In this lesson, we will also see that it is possible to cancel dimensions and units.

Dimensional Analysis

If x is a measurement of distance and t is a measurement of time, then what does $\frac{x}{t}$ represent physically?

We will explore the answers to this question in this section.

Dimension

We can use math to describe many physical things. Therefore, it is helpful to define the physical nature of a mathematical object. The **dimension** of a variable or number is a property that tells us what *type* of physical quantity it represents. For example, some possible dimensions are

Length Time Mass Speed Force Energy

In our opening question, x is a measurement of distance. Therefore, x has the dimension of

______________.

Note that some variables and numbers have no relation to anything physical. These quantities are called ________________________.
These quantities usually occur as a result of multiplying or dividing quantities with dimensions such that the dimensions cancel each other out. For example, comparing the height of a box to its width would give you,

$$\frac{height}{width}$$

When considering their dimensions,

$$\frac{length}{length} = 1$$

Notice there are no more dimensions remaining, thus $\frac{height}{width}$ is a dimensionless quantity.

Notation

We write the dimension of a variable, say x, as $[x]$. Therefore, $[x] =$ ______________

Fundamental/Base Dimensions

Dimensions are related to each other in natural ways. For example, it is natural to think of speed as the ratio of length to time since your speed depends on how far you travel and how long it takes you to go that far. That is,

$$\text{speed} = \frac{\text{length}}{\text{time}}$$

It turns out that all of the dimensions we know can be derived from just seven dimensions through multiplication and division. These seven dimensions are called **fundamental dimensions** or **base dimensions**. Dimensions composed of these seven dimensions are called **derived dimensions**.

It is interesting to note that there is more than one correct choice for the seven base dimensions. However, to keep things the same, the International System of Units (SI) has picked seven dimensions to be the standard base dimensions:

$$\begin{aligned}
&\text{Length} \rightarrow L \\
&\text{Time} \rightarrow T \\
&\text{Mass} \rightarrow M \\
&\text{Electric Current} \rightarrow I \\
&\text{Absolute Temperature} \rightarrow \Theta \\
&\text{Amount of Substance} \rightarrow N \\
&\text{Luminous Intensity} \rightarrow J
\end{aligned}$$

They are chosen due to their natural link to fundamental constants of the universe (pretty cool!). Because we use base dimensions so much, we have short forms for writing them (as seen above).

Rules for Dimensional Quantities

1. Only variables and/or numbers of the ______________ dimension may be compared, equated, added, or subtracted.
2. Variables and/or numbers of ______________ dimension may be multiplied, divided, or expressed as rates or ratios.
3. Dimensional quantities may be multiplied or divided by ________________________ quantities. The resulting quantity has the same dimension as the original dimensional quantity.

Notes:

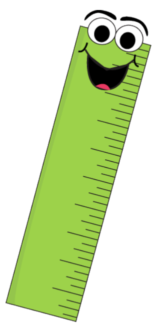
- Rule _____ helps us to avoid calculations that don't make sense. For example, "What is 3 metres plus 1 hour?"
- Rule _____ allows us to derive dimensions from the base dimensions and to have expressions and equations with variables/numbers of more than one dimension.
- Rule _____ allows us to have expressions that involve both dimensionless and dimensional quantities.

Unit Analysis

Units of Measurement

A **unit of measurement** is a standard amount of a physical quantity. It is not to be confused with the idea of dimension. In fact, a variable of a certain dimension can be measured in many different units of measurement.

e.g. Think of length again. It can be measured in centimetres, kilometres, feet, inches, etc.



A ruler is used to make measurements of the dimension length using different units of measurement such as centimetres, millimetres, or inches.

Exercise. Name multiple units for each dimension.

Time:

Mass:

Metric Prefixes

In the metric system, prefixes are added to units of measurement in order to express very large or very small numbers more easily.

Below are some of the common prefixes used in the metric system:

Prefix	**Symbol**	**Prefix as a Decimal**
Giga	G	$1,000,000,000$
Mega	M	$1,000,000$
Kilo	k	1000
Hecto	h	100
Deca	da	10
(none)	(none)	1
Deci	d	0.1
Centi	c	0.01
Milli	m	0.001
Micro	μ	0.000001
Nano	n	0.000000001

Using the chart, we can see that $1cm = 0.01m$. Or in other words, one centimetre is one hundredth of a metre.

Below are the standard units for some of the most common dimensions:

Dimension	**Standard Unit**	**Symbol**
Length	Metre	m
Time	Second	s
Mass	Gram	g
Electric Current	Ampere	A
Absolute Temperature	Kelvin	K
Amount of Substance	Mole	mol
Luminous Intensity	Candela	cd
Speed	Metres Per Second	m/s
Acceleration	Metres Per Second Squared	m/s^2
Force	Newton	$\text{N} = \text{kg} \times \text{m/s}^2$
Energy	Joule	J

Unit Conversions in the Metric System

It is easy to convert between units of the same dimension in the metric system. To make conversions, we will make use of unit identities and the standard unit of the dimension.

A **unit identity** is a quotient of units of the same dimension that equals the dimensionless value of 1. For example,

$$\frac{1 \text{ hm}}{100 \text{ m}} = 1$$

Examples

1. Since 1 kg = 1000 g,

$$6 \text{ g} = (6 \text{ g})\left(\frac{1 \text{ kg}}{1000 \text{ g}}\right) = (6 \cancel{\text{g}})\left(\frac{1 \text{ kg}}{1000 \cancel{\text{g}}}\right) = (6)\left(\frac{1 \text{ kg}}{1000}\right) = \frac{6}{1000} \text{ kg} = 0.006 \text{ kg}$$

2. Since 1 ms = 0.001 s,

$$2 \text{ ms} = (2 \text{ ms})\left(\frac{0.001 \text{ s}}{1 \text{ ms}}\right) = (2 \cancel{\text{ms}})\left(\frac{0.001 \text{ s}}{1 \cancel{\text{ms}}}\right) = (2)(0.001 \text{ s}) = 0.002 \text{ s}$$

3. Since 1 μA = 0.000001 A, then 1000000 μA = 1 A. Thus,

$$7 \text{ A} = (7 \text{ A})\left(\frac{1000000 \ \mu\text{A}}{1 \text{ A}}\right) = (7 \cancel{\text{A}})\left(\frac{1000000 \ \mu\text{A}}{1 \cancel{\text{A}}}\right) = (7)(1000000 \ \mu\text{A}) = 7000000 \ \mu\text{A}$$

In general, follow the steps below to convert between units of the same dimension:

1. Write the quantity in the original units (as given).
2. Multiply the quantity by the unit identity involving the original unit and the standard unit so as to cancel the original units. Use the prefix chart to help you!
3. Multiply the result from step 2 by the unit identity involving the desired unit and the standard unit so as to cancel the standard unit. Use the prefix chart to help you!

Note: If the standard unit is the desired unit, you are finished after step 2.

Time Calculations

Time is the most commonly used dimension. Most people do time calculations every day using time.
Test your knowledge of the units of time by filling out the table below.

seconds	minutes	hours	days	weeks
1				
	1			
		1		
			1	
				1
	660			
				4
			31	
31449600				
	90720			

Problem Set

1. Fill in the blanks.

   (a) 1 ng = ____________ g

   (b) 5 ks = ____________ ms

   (c) 1000 MJ = ____________ GJ

   (d) 10 hK = ____________ daK

2. How many seconds are there in a leap year?

3. Which of the following expressions/statements do not make sense?

   (a) $x + y = z$ where $[x] = [y] \neq [z]$ (The dimension of x is the same as the dimension of y which is different from the dimension of z)

   (b) $x - y$ where $[x] = [y]$

   (c) $\frac{xz}{y}$ where $[x] \neq [y] \neq [z]$

   (d) The length of the fence is greater than its mass.

   (e) The ratio of the length of fence A to its mass is greater than the ratio of the length of fence B to its mass.

4. The product of $60 \times 60 \times 24 \times 7$ equals which of the following?
   (a) the number of minutes in seven weeks
   (b) the number of hours in sixty days
   (c) the number of seconds in seven hours
   (d) the number of seconds in one week
   (e) the number of minutes in twenty-four weeks
5. Find the correct value for all of the choices in question 4.

CHALLENGE

6. A bamboo plant grows at a rate of 105cm per day. On May 1st at noon it was 2m tall. Approximately how tall, in meters, was it on May 8th at noon?

7. Jessica is throwing a pizza party! There are 20 people including herself that are expected at the party. Jessica predicts that each person will eat 3 slices of pizza and drink 1 pop.
   Pop comes in cases of 12 for \$10 a case, or individually at \$1.25 per can.
   Large pizzas have 10 slices and cost \$12. Extra-large pizzas have 12 slices and cost \$14.
   What is the minimum amount of money that Jessica can spend and still ensure that everyone at the party gets their share of pizza and pop?

8. Owen spends \$1.20 per litre on gasoline. He uses an average of 1L of gasoline to drive 12.5 km. How much will Owen spend on gasoline to drive 50 km?

9. Anca and Bruce left Mathville at the same time. They drove along a straight highway towards Staton. Bruce drove at 50 km/h. Anca drove at 60 km/h, but stopped along the way to rest. They both arrived at Staton at the same time. For how long did Anca stop to rest?