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Mathematics and Computing

Grade 6 Math Circles

February 26th/27th

Units and Dimensional Analysis

Introduction

Cancelling is the removal of a common factor in the numerator (top) and denominator (bottom) of a fraction. Here is an example of cancelling:

$$(2)\left(\frac{8}{10}\right) = \frac{(2)(8)}{10} = \frac{(2)(8)}{(2)(5)} = \frac{\cancel{(2)}(8)}{\cancel{(2)}(5)} = \frac{8}{5}$$

We can also think of cancelling as recognizing when an operation is undone by an opposite operation. The most common example we will see are the opposite operations of multiplication and division. In the example above, the division by 2 is undone by the multiplication by 2, so we can cancel the factors of 2. If we skip writing out all the steps, we could have shown the cancelling above as follows:

$$(2)\left(\frac{8}{10}\right) = \cancel{(2)}\left(\frac{8}{\cancel{(2)}(5)}\right) = \frac{8}{5}$$

We can also cancel variables in the same way we cancel numbers. In this lesson, we will also see that it is possible to cancel dimensions and units.

Dimensional Analysis

If x is a measurement of distance and t is a measurement of time, then what does $\frac{x}{t}$ represent physically?

We will explore the answers to this question in this section.

Dimension

We can use math to describe many physical things. Therefore, it is helpful to define the physical nature of a mathematical object. The **dimension** of a variable or number is a property that tells us what *type* of physical quantity it represents. For example, some possible dimensions are

Length Time Mass Speed Force Energy

In our opening question, x is a measurement of distance. Therefore, x has the dimension of length.

Note that some variables and numbers have no relation to anything physical. These quantities are called dimensionless.
These quantities usually occur as a result of multiplying or dividing quantities with dimensions such that the dimensions cancel each other out. For example, comparing the height of a box to its width would give you,

$$\frac{height}{width}$$

When considering their dimensions,

$$\frac{length}{length} = 1$$

Notice there are no more dimensions remaining, thus $\frac{height}{width}$ is a dimensionless quantity.

Notation

We write the dimension of a variable, say x, as $[x]$. Therefore, $[x] =$ length

Fundamental/Base Dimensions

Dimensions are related to each other in natural ways. For example, it is natural to think of speed as the ratio of length to time since your speed depends on how far you travel and how long it takes you to go that far. That is,

$$\text{speed} = \frac{\text{length}}{\text{time}}$$

It turns out that all of the dimensions we know can be derived from just seven dimensions through multiplication and division. These seven dimensions are called **fundamental dimensions** or **base dimensions**. Dimensions composed of these seven dimensions are called **derived dimensions**.

It is interesting to note that there is more than one correct choice for the seven base dimensions. However, to keep things the same, the International System of Units (SI) has picked seven dimensions to be the standard base dimensions:

$$\begin{aligned}
&\text{Length} \rightarrow L \\
&\text{Time} \rightarrow T \\
&\text{Mass} \rightarrow M \\
&\text{Electric Current} \rightarrow I \\
&\text{Absolute Temperature} \rightarrow \Theta \\
&\text{Amount of Substance} \rightarrow N \\
&\text{Luminous Intensity} \rightarrow J
\end{aligned}$$

They are chosen due to their natural link to fundamental constants of the universe (pretty cool!). Because we use base dimensions so much, we have short forms for writing them (as seen above).

Rules for Dimensional Quantities

1. Only variables and/or numbers of the same dimension may be compared, equated, added, or subtracted.

2. Variables and/or numbers of any dimension may be multiplied, divided, or expressed as rates or ratios.

3. Dimensional quantities may be multiplied or divided by dimensionless quantities. The resulting quantity has the same dimension as the original dimensional quantity.

Notes:

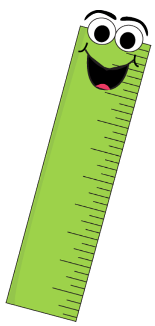
- Rule 1 helps us to avoid calculations that don't make sense. For example, "What is 3 metres plus 1 hour?"
- Rule 2 allows us to derive dimensions from the base dimensions and to have expressions and equations with variables/numbers of more than one dimension.
- Rule 3 allows us to have expressions that involve both dimensionless and dimensional quantities.

Unit Analysis

Units of Measurement

A **unit of measurement** is a standard amount of a physical quantity. It is not to be confused with the idea of dimension. In fact, a variable of a certain dimension can be measured in many different units of measurement.

e.g. Think of length again. It can be measured in centimetres, kilometres, feet, inches, etc.



A ruler is used to make measurements of the dimension length using different units of measurement such as centimetres, millimetres, or inches.

Exercise. Name multiple units for each dimension.

Time:

Second, minute, hour, day, week, month, year, millisecond, nanosecond, etc.

Mass:

Gram, kilogram, milligram, pound, ton, tonne, etc.

Metric Prefixes

In the metric system, prefixes are added to units of measurement in order to express very large or very small numbers more easily.

Below are some of the common prefixes used in the metric system:

Prefix	**Symbol**	**Prefix as a Decimal**
Giga	G	$1,000,000,000$
Mega	M	$1,000,000$
Kilo	k	1000
Hecto	h	100
Deca	da	10
(none)	(none)	1
Deci	d	0.1
Centi	c	0.01
Milli	m	0.001
Micro	μ	0.000001
Nano	n	0.000000001

Using the chart, we can see that $1cm = 0.01m$. Or in other words, one centimetre is one hundredth of a metre.

Below are the standard units for some of the most common dimensions:

Dimension	**Standard Unit**	**Symbol**
Length	Metre	m
Time	Second	s
Mass	Gram	g
Electric Current	Ampere	A
Absolute Temperature	Kelvin	K
Amount of Substance	Mole	mol
Luminous Intensity	Candela	cd
Speed	Metres Per Second	m/s
Acceleration	Metres Per Second Squared	m/s^2
Force	Newton	$\text{N} = \text{kg} \times \text{m/s}^2$
Energy	Joule	J

Unit Conversions in the Metric System

It is easy to convert between units of the same dimension in the metric system. To make conversions, we will make use of unit identities and the standard unit of the dimension.

A **unit identity** is a quotient of units of the same dimension that equals the dimensionless value of 1. For example,

$$\frac{1\ \text{hm}}{100\ \text{m}} = 1$$

Examples

1. Since 1 kg = 1000 g,

$$6\ \text{g} = (6\ \text{g})\left(\frac{1\ \text{kg}}{1000\ \text{g}}\right) = (6\ \cancel{\text{g}})\left(\frac{1\ \text{kg}}{1000\ \cancel{\text{g}}}\right) = (6)\left(\frac{1\ \text{kg}}{1000}\right) = \frac{6}{1000}\ \text{kg} = 0.006\ \text{kg}$$

2. Since 1 ms = 0.001 s,

$$2\ \text{ms} = (2\ \text{ms})\left(\frac{0.001\ \text{s}}{1\ \text{ms}}\right) = (2\ \cancel{\text{ms}})\left(\frac{0.001\ \text{s}}{1\ \cancel{\text{ms}}}\right) = (2)(0.001\ \text{s}) = 0.002\ \text{s}$$

3. Since 1 μA = 0.000001 A, then 1000000 μA = 1 A. Thus,

$$7\ \text{A} = (7\ \text{A})\left(\frac{1000000\ \mu\text{A}}{1\ \text{A}}\right) = (7\ \cancel{\text{A}})\left(\frac{1000000\ \mu\text{A}}{1\ \cancel{\text{A}}}\right) = (7)(1000000\ \mu\text{A}) = 7000000\ \mu\text{A}$$

In general, follow the steps below to convert between units of the same dimension:

1. Write the quantity in the original units (as given).
2. Multiply the quantity by the unit identity involving the original unit and the standard unit so as to cancel the original units. Use the prefix chart to help you!
3. Multiply the result from step 2 by the unit identity involving the desired unit and the standard unit so as to cancel the standard unit. Use the prefix chart to help you!

Note: If the standard unit is the desired unit, you are finished after step 2.

Time Calculations

Time is the most commonly used dimension. Most people do time calculations every day using time.

Test your knowledge of the units of time by filling out the table below.

seconds	**minutes**	**hours**	**days**	**weeks**
1				
	1			
		1		
			1	
				1
	660			
				4
			31	
31449600				
	90720			

seconds	**minutes**	**hours**	**days**	**weeks**
1	$\frac{1}{60} = 0.1\overline{6}$	$\frac{1}{3600} = 0.0002\overline{7}$	$\frac{1}{86400} = 0.0000115\overline{740}$	$\frac{1}{604800} = 0.0000016\overline{534391}$
60	1	$\frac{1}{60} = 0.01\overline{6}$	$\frac{1}{1440} = 0.00069\overline{4}$	$\frac{1}{10080} = 0.0000\overline{9920634}$
3600	60	1	$\frac{1}{24} = 0.041\overline{6}$	$\frac{1}{168} = 0.005\overline{952380}$
86400	1440	24	1	$\frac{1}{7} = 0.\overline{142857}$
604800	10080	168	7	1
39600	660	11	$\frac{11}{24} = 0.458\overline{3}$	$\frac{11}{168} = 0.065\overline{476190}$
2419200	40320	672	28	4
2678400	44640	744	31	$\frac{31}{7} = 4.\overline{428571}$
31449600	524160	8736	364	52
5443200	90720	1512	63	9

1 **second** compared to 1 **week** is a similar size difference as comparing the length of a human to a single red blood cell

1 **second** compared to 1 **day** is a similar size difference as comparing the length of a human to the width of a human hair.

1 **second** compared to 1 **hour** is a similar size difference as comparing the length of a human to single pixel in a computer monitor.

1 **second** compared to 1 **minute** is a similar size difference as comparing the length of a human to large apple.

Problem Set

1. Fill in the blanks.

   (a) 1 ng = ____________ g

   0.000000001 g

   $$1 \text{ ng} = (1 \cancel{\text{ng}}) \left(\frac{1 \text{ g}}{1000000000 \cancel{\text{ng}}} \right) = (0.000000001\text{g})$$

   (b) 5 ks = ____________ ms

   5 000 000 ms

   $$5 \text{ ks} = (5 \cancel{\text{ks}}) \left(\frac{1 \cancel{\text{s}}}{1000 \cancel{\text{ks}}} \right) \left(\frac{1 \text{ ms}}{100 \cancel{\text{s}}} \right) = (5000000\text{ms})$$

   (c) 1000 MJ = ____________ GJ

   1 GJ

   $$1000 \text{ MJ} = (1000 \cancel{\text{MJ}}) \left(\frac{1000000 \cancel{\text{J}}}{1 \cancel{\text{MJ}}} \right) \left(\frac{1 \text{ GJ}}{1000000000 \cancel{\text{J}}} \right) = (1\text{GJ})$$

   (d) 10 hK = ____________ daK

   100 daK

   $$10 \text{ hK} = (10 \cancel{\text{hK}}) \left(\frac{100 \cancel{\text{K}}}{1 \cancel{\text{hK}}} \right) \left(\frac{1 \text{ daK}}{10 \cancel{\text{K}}} \right) = (100\text{daK})$$

2. How many seconds are there in a leap year?

   $$1\text{leap year} \times 366 \frac{\text{days}}{\text{leap year}} \times 24 \frac{\text{hours}}{\text{day}} \times 60 \frac{\text{minutes}}{\text{hour}} \times 60 \frac{\text{seconds}}{\text{minute}} = 31622400 \text{ seconds}$$

   31 622 400 seconds are in a leap year

3. Which of the following expressions/statements do not make sense?

   (a) $x + y = z$ where $[x] = [y] \neq [z]$ (The dimension of x is the same as the dimension of y which is different from the dimension of z)

   (b) $x - y$ where $[x] = [y]$

   (c) $\frac{xz}{y}$ where $[x] \neq [y] \neq [z]$

   (d) The length of the fence is greater than its mass.

   (e) The ratio of the length of fence A to its mass is greater than the ratio of the length of fence B to its mass.

   (a) does not make sense as you cannot add the same dimension to produce a different dimension.

   (d) does not make sense as you cannot compare different dimensions.

4. The product of $60 \times 60 \times 24 \times 7$ equals which of the following?

   (a) the number of minutes in seven weeks

   (b) the number of hours in sixty days

   (c) the number of seconds in seven hours

   (d) the number of seconds in one week

   (e) the number of minutes in twenty-four weeks

   (d) the number of seconds in one week

   $60\frac{\text{seconds}}{\text{minute}} \times 60\frac{\text{minutes}}{\text{hours}} \times 24\frac{\text{hours}}{\text{day}} \times 7\frac{\text{days}}{\text{week}}$

5. Find the correct value for all of the choices in question 4.

   (a) $60 \times 24 \times 7 \times 7$

   $60\frac{\text{minutes}}{\text{hour}} \times 24\frac{\text{hours}}{\text{day}} \times 7\frac{\text{days}}{\text{week}} \times 7$ weeks

   (b) 24×60

   $24\frac{\text{hours}}{\text{day}} \times 60$ days

   (c) $60 \times 60 \times 7$

   $60\frac{\text{seconds}}{\text{minute}} \times 60\frac{\text{minutes}}{\text{hour}} \times 7$ hours

   (e) $60 \times 24 \times 7 \times 24$

   $60\frac{\text{minutes}}{\text{hours}} \times 24\frac{\text{hours}}{\text{day}} \times 7\frac{\text{days}}{\text{week}} \times 24$ weeks

CHALLENGE

6. A bamboo plant grows at a rate of 105cm per day. On May 1st at noon it was 2m tall. Approximately how tall, in meters, was it on May 8th at noon?

   9.35 m

   Since the bamboo plant grows at a rate of 105cm per day and there are 7 days from May 1st to May 8th, then it grows 7 days $\times$ $105\frac{\text{cm}}{\text{day}}$ = 735 cm in this time period. Since 735 cm = 7.35 m, then the height of the plant on May 8th is 2 m + 7.35 m = 9.35 m

   2005 Gauss (Grade 7) #12

7. Jessica is throwing a pizza party! There are 20 people including herself that are expected at the party. Jessica predicts that each person will eat 3 slices of pizza and drink 1 pop.
Pop comes in cases of 12 for \$10 a case, or individually at \$1.25 per can.
Large pizzas have 10 slices and cost \$12. Extra-large pizzas have 12 slices and cost \$14.
What is the minimum amount of money that Jessica can spend and still ensure that everyone at the party gets their share of pizza and pop?
\$90

To solve the question, we really have to solve two smaller questions:

(1) What is the minimum cost of giving 20 people one pop each?

(2) What is the minimum cost of giving 20 people 3 slices of pizza each?

Let's solve question (1) first:
Since each person wants one pop, Jessica must buy at least 20 pops to ensure everyone at the party gets a pop.
We can think of a *case* as a unit and a *pop* as a unit. Then 1 case = 12 pop. Clearly, one case of pop is not enough since 12 is less than 20. However, 2 cases is enough:

$$2 \text{ case} = (2 \cancel{\text{case}})\left(\frac{12 \text{ pop}}{1 \cancel{\text{case}}}\right) = 24 \text{ pop}$$

The cost of 2 cases of pop is

$$(2 \cancel{\text{case}})\left(\frac{\$10}{1 \cancel{\text{case}}}\right) = \$20$$

If Jessica buys 20 cans of pop instead, it will cost her

$$(20 \cancel{\text{pop}})\left(\frac{\$1.25}{1 \cancel{\text{pop}}}\right) = \$25$$

<u>Note:</u> If Jessica buys 1 case of pop and the remaining 8 cans individually, it will cost her

$$(1 \cancel{\text{case}})\left(\frac{\$10}{1 \cancel{\text{case}}}\right) + (8 \cancel{\text{pop}})\left(\frac{\$1.25}{1 \cancel{\text{pop}}}\right) = \$10 + \$10 = \$20$$

This is the same amount as buying 2 cases.

Therefore, the minimum amount of money Jessica can spend on pop is \$20.

Now to solve question (2):

We will begin to think of person as a unit of people, and large, xlarge, and slice as units of pizza as well. Since there are 10 slices in a large pizza, 1 large = 10 slice. Similarly, 1 xlarge = 12 slice.

The total amount of pizza Jessica needs to buy is

$$(20\ \cancel{\text{person}})\left(\frac{3\ \text{slice}}{1\ \cancel{\text{person}}}\right) = 60\ \text{slice}$$

Now using the idea of unit identities, we can get that

$$60\ \text{slice} = (60\ \cancel{\text{slice}})\left(\frac{1\ \text{large}}{10\ \cancel{\text{slice}}}\right) = 6\ \text{large}$$

and that

$$60\ \text{slice} = (60\ \cancel{\text{slice}})\left(\frac{1\ \text{xlarge}}{12\ \cancel{\text{slice}}}\right) = 5\ \text{xlarge}$$

That is, Jessica can either buy 6 large pizzas or 5 extra large pizzas to make sure that everybody has 3 slices of pizza.

The cost of 6 large pizzas is

$$(6\ \cancel{\text{large}})\left(\frac{\$12}{1\ \cancel{\text{large}}}\right) = \$72$$

The cost of 5 extra large pizzas is

$$(5\ \cancel{\text{xlarge}})\left(\frac{\$14}{1\ \cancel{\text{xlarge}}}\right) = \$70$$

Clearly 70 is less than 72, so the minimum amount of money Jessica can spend on pizza is \$70.

Putting the answers of (1) and (2) together, we get that the minimum amount of money that Jessica can spend (so that everyone gets their share of pizza and pop) is $\$20 + \$70 = \$90$.

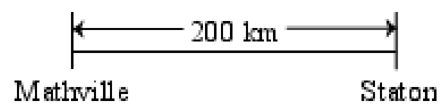
Aside: An Interesting Note

In this solution, I made up "units" like *person* or *slice*. Really, all of these quantities can be argued to have the dimension of *amount of substance*, N. But aren't we just counting? Shouldn't all these values really be dimensionless? This is a subtle point, but there is a difference between the ideas of N and a dimensionless quantity. When a number or variable has the dimension N, we know that the number or variable corresponds to a counted number of *physical objects* of some substance. This substance could be oxygen, or it could be pizza. Meanwhile, dimensionless quantities are pure numbers. Dimensionless counting is like counting without objects (it is an abstract idea). To illustrate the difference between N and dimensionless numbers, consider multiplying a length by 2 versus multiplying the length by 2 pizzas (do you see the difference?).

8. Owen spends \$1.20 per litre on gasoline. He uses an average of 1L of gasoline to drive 12.5 km. How much will Owen spend on gasoline to drive 50 km?

   \$ 4.80
   2013 Pascal Contest #9.
   https://www.cemc.uwaterloo.ca/contests/past_contests.html for full solutions.

9. Anca and Bruce left Mathville at the same time. They drove along a straight highway towards Staton. Bruce drove at 50 km/h. Anca drove at 60 km/h, but stopped along the way to rest. They both arrived at Staton at the same time. For how long did Anca stop to rest?



   40 minutes
   2015 Fermat Contest #10.
   https://www.cemc.uwaterloo.ca/contests/past_contests.html for full solutions.