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# Grade 6 Math Circles

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***Exponentiation***

## Powers

When you first learned to multiply, you probably learned to think of it as repeated addition; for example, $4 \times 3$ = adding four three times (or adding three four times) $= 4+4+4 = 12$. In much the same way, we can introduce a new operation which is like repeated multiplication. For example, if we wanted to express four times itself three times, $4 \times 4 \times 4$, we could write it like this: $4^3$. We call this operation ________________, and we represent it like this: $\text{base}^{\text{exponent}}$. When reading this notation aloud, we say "(base) to the power of (exponent)," or sometimes just "(base) to the (exponent)." An important thing to notice is that this is not like multiplication in the sense that *order matters.* For example, $9^2 = 81 \neq 2^9 = 512$. In general, for two numbers $a$ and $b$, $a^b \neq b^a$.

**Try it yourself #1**

Evaluate the following:

1. $4^3$
2. $2^5$
3. $8^2$

You may already have questions such as, "if we can exponentiate with any numbers, what about zero?" or, "what about one?" or even, "what about negative numbers?" Let's think about the answers to those questions.

**Try it yourself #2**

To answer some of the above questions, attempt the following:

1. $1^2$
2. $1^3$
3. $1^{2019}$
4. $0^2$
5. $0^3$
6. $0^{2019}$

As you can see, if 1 is our base, then we can multiply it by itself as many times as we like and we'll always get 1. The same is true of 0.

**Try it yourself #3**

Evaluate:

1. $2^1$
2. $3^1$
3. $2019^1$

If one is our *exponent*, then we just get our base back – multiplying a number once gives the number we started with. Zero as an exponent is a little strange, so we'll come back to it later. Negative exponents are also a little strange, but we'll go through them now and explain them later. A number raised to a negative power is simply the **reciprocal** of the number raised to that power. The reciprocal of a number is 1 divided by that number. In symbols, where $a$ is a number and $x$ is a positive number, $a^{-x} = \frac{1}{a^x}$. For example, the reciprocal of 5 is $\frac{1}{5}$. In exponential notation, we can say that the reciprocal of 5 is just $5^{-1}$.

**Try it yourself #4**

Evaluate (with a calculator if necessary) or at least simplify:

1. $2^{-3}$
2. $5^{-2}$
3. $10^{-4}$

We can use exponentiation with any numbers we like, it just becomes a little more difficult to think about. For example, if you try $4.1^{3.3}$ on your calculator, you'll get roughly 105.241426756. How did the calculator do this? To understand, we have to talk a bit more about the exponential.

# Roots

First, however, we are going to talk about roots. You have all seen the symbol, "$\sqrt{\text{number}}$" before. This symbol denotes the ____________ and means "what number, when I multiply it by itself (*square* it), gives me my original number?" For example, $3 \times 3 = 9$ so $3 = \sqrt{9}$. (Observe that $-3 \times -3 = 9$ as well). Notice also that we cannot take the square root of a negative number—no real number times itself is negative! We can extend the question we asked earlier—that is, make it more general—and ask instead, "what number do I have to multiply by itself $n$ times to get my original number," where $n$ is just a whole number.

We write that question down as, "$\sqrt[n]{\text{number}} = ?$" and call the answer the $n^{\text{th}}$ root of the number.

**Try it yourself #5**
Evaluate the following roots:

1. $\sqrt{49}$,
2. $\sqrt{81}$,
3. $\sqrt[3]{-27}$,
4. $\sqrt[5]{32}$.

Again, we can take any number to any power we like. This means that we can, for example, take $4^{\frac{1}{2}} = 2$.

**Try it yourself #6**
Evaluate the following with a calculator, if necessary:

1. $49^{\frac{1}{2}}$
2. $81^{\frac{1}{2}}$
3. $(-27)^{\frac{1}{3}}$
4. $32^{\frac{1}{5}}$

Notice anything? This is because $\sqrt[n]{\ }$ and $\frac{1}{n}$ are actually two different ways to say the same thing! It turns out that we can write powers and roots interchangeably. This is due to the following property of exponents:

**Property #1:** For any real number $a$ and any two exponents, $x$ and $y$, $a^x \times a^y = a^{x+y}$.

**Try it yourself #7**
Evaluate the following:

1. $2^2 \times 2^2$
2. $3^2 \times 3$
3. $5^2 \times 5^2$
4. $6^1 \times 6^{-1}$

We're finally in a position to understand what fractions and zero do as exponents. To understand zero, recall that any number times its reciprocal is one, *i.e.* $a \times \frac{1}{a} = 1$. We can write this in exponential notation as $a^1 \times a^{-1} = 1$. We know the property of exponents that we discussed earlier, so we can see that $a^1 \times a^{-1} = a^{1-1} = a^0$. So $a^0$ should be ___. (This is true in every case except when $a = 0$. Things with zero can be weird sometimes.)

To understand roots, recall that $\frac{1}{2} + \frac{1}{2} = 1$ What if we combine this with the first exponent property? We can see that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$. Does this look familiar? It's kind of hard to see—disguised, almost—but when we ask "what number multiplied by itself gives me my starting number," this is another way to think about it. The same idea applies to third, fourth, and higher roots, for example, $2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{4}} = 2$.

For fractions as the base, we simply take the numerator to the given exponent, and the denominator to the given exponent. For example, $\left(\frac{5}{4}\right)^2 = \frac{5^2}{4^2} = \frac{25}{16}$.

So far, we've only seen reciprocals of whole numbers. What about other fractions? To learn about these, we'll introduce another property of exponents.

**Property #2:** For any non-negative base $a$ and any two exponents, $s$ and $t$, $(a^s)^t = a^{s \times t}$.

Now if we wanted to figure out what $8^{\frac{2}{3}}$ is, we can first "reverse-engineer" the exponent into $8^{2 \times \frac{1}{3}}$. Now we can recognize this as $\sqrt[3]{8^2}$ and simplify: ______.

**Try it yourself #8**

Simplify and evaluate the following if able:

1. $(2^2)^2$
2. $(3^4)^{\frac{1}{2}}$
3. $(5^2)^{-1}$
4. $(6^1)^{-2}$

# Problems

1. Evaluate the following:
   (a) $5^2$
   (b) $3^3$
   (c) $4^{-1}$
   (d) $9^{\frac{1}{2}}$
   (e) $64^{-\frac{1}{3}}$
   (f) $8^{-\frac{2}{3}}$

2. Use the first property of exponents to simplify:
   (a) $2^2 \times 2^3$
   (b) $3^8 \times 3^{-6}$
   (c) $\sqrt{5} \times 5^{\frac{3}{2}}$
   (d) $8^{19} \times \frac{1}{8^{20}}$
   (e) $11^2 \times \frac{1}{\sqrt[3]{11}} \div 11^{-\frac{2}{3}}$

3. Use the second property of exponents to simplify:
   (a) $(2^2)^{\frac{1}{2}}$
   (b) $(3^{-1})^{-3}$
   (c) $\left(\left(\frac{1}{5}\right)^2\right)^{-\frac{3}{2}}$
   (d) $\left(8^{\frac{1}{3}}\right)^{\sqrt[3]{27}}$
   (e) $\left(\left(\frac{1}{\sqrt{11}}\right)^{\frac{1}{4}}\right)^{-\sqrt{16}}$

4. Solve for $x$:
   (a) $10^x - 10 = 9990$
   (b) $4^x = 64^2$
   (c) $3^6 = 27^x$
   (d) $5^{33} = 125^x$
   (e) $6^{x+2} = 216$
   (f) $8^{x-1} = 2^6$

5. Polynomials: a polynomial is an expression involving $x$ and its various powers. Polynomials are very important in math because they are easy to work with and can be used to study and model a large number of things very accurately. So what do they look like? They look like

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

That probably looks a little confusing, so let's break that down. Here, $n$ is a whole number. It can be as low as 1 or as big as we want. The little $n$ beside the $a$ is called a **subscript**, and it just means that $a_n$ goes with $x^n$. It's for the sake of organization. $a_n$ represents a number that is being multiplied to $x$. $x$ is **arbitrary**, that is, it can be any number we want it to be. Finally, what about the dots? In math, those dots mean "and so on," and let us represent things **in general**, rather than for a specific size. There can be anywhere from no terms to as many terms as we want hidden by the dots, as long as we give enough that we can tell what the pattern is. As we talked about before, we can do a ton with polynomials. However, before we run, we have to walk. Start by evaluating the following:

   (a) $x^2 - 9$, where $x = -3$
   (b) $x^2 - 2x + 1$, where $x = 2$
   (c) $x^2 + 4x - 5$, where $x = 4$
   (d) $x^3 + x^2 + x + 1$, where $x = -1$
   (e) $3x^4 - 5x^3 + x^2 + 12x - 8$, where $x = 0$
   (f) $2x^5 - 3x^4 + 31x^3 - 8x^2 + x - 10$, where $x = 5$

6. **Challenge:** Find two distinct (non-equal) positive integers, $x$ and $y$, such that $x^y = y^x$.