UNIVERSITY OF
WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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Counting

Introduction

You might be scratching your head right now wondering why our topic today is **counting**! Surely everyone here already knows how to count right? Are we going to spend the entire 1.5 hours of Math Circles counting from 1 to 1,000,000,000?!

What we will learn today is how to count the number of different possible outcomes of specific actions and events. This type of mathematical counting is the foundation of a entire branch of math called **combinatorics**. How is this different from counting 1, 2, 3,. . . ? Well, we will learn mathematical "shortcuts" to counting up to very large numbers quickly and accurately!

How many choices?

Exercise: Ollivander's Wand Making

Ollivander is making a new wand. These are the choices that he can make for the wand's length, wood, and core:

- **Length**: $9\frac{3}{4}$ inches or $15\frac{1}{4}$ inches
- **Wood**: Hawthorne or Maple
- **Core**: Unicorn hair, Dragon heartstring, or Phoenix feather

How many different types of wands can Ollivander make? Draw a tree diagram to show your work on the next page.

Fundamental Counting Principle

If you have to make Choice A **AND** Choice B, and there are m options for Choice A and n options for Choice B, then the total number of different ways you can make Choice A and Choice B is $m \times n$.

When you see "**AND**" in a counting problem, it is a hint that you have to use this rule! Let's apply this rule to the Ollivander's Wand Making problem.

Exercise: Ollivander's Wand Making continued

How many choices do we have to make?

How many options are there for each choice?

Use the Fundamental Counting Principle to find the total number of different wands that Ollivander can make. Check that this is the same answer you got with the tree diagram before.

Exercise: Luna's Outfit
Luna Lovegood is picking her outfit for the day. She needs to pick a top, bottom, accessory, and shoes. Here's what Luna has in her wardrobe:

- **Top**: Nargle infested sweater or Ravenclaw sweater
- **Bottom**: Paint splattered skirt, blue wool leggings, or red shorts
- **Accessory**: Butterbeer cork necklace, plum earrings, Spectrespecs, lion hat, eagle hat, or Ravenclaw scarf
- **Shoes**: Loafers, sneakers, or high heels

How many choices does Luna have to make?

How many different outfits could Luna put together?

How many orders?

Exercise: Potions Problem
Professor Snape gives Harry, Hermione, Draco, and Pansy an extra essay assignment as punishment for fighting in Potions class! Each student could get one of 6 grades: Outstanding, Exceeds Expectations, Acceptable, Poor, Dreadful, or Troll. Snape will then write the grades on the board because he's just that mean!

Is repetition allowed in the outcome (the grades that Snape writes on the board)?

How many different outcomes are there? (i.e. How many different ways can these students be graded?) Use Fundamental Counting Principle.

Exercise: Professors Wanted

Dumbledore is interviewing four candidates: Galatea Merrythought, Septima Vector, Vindictus Viridian, and Herbert Beery. All four of these professors are qualified to teach Potions, Arithmancy, Herbology, and Defense Against the Dark Arts (DADA) but each can only be assigned to one subject.

Is repetition allowed?

How many different ways can Dumbledore match up the professors with the classes?

Factorials

Whenever we have a list or group of n things and we need to figure out how many different orders they can be put in (how many different shuffles) with **no repetition**, we use the same type of logic that we just applied to the previous problem. This will always result in the answer being $n \times (n-1) \times (n-2) \times (n-3) \times ... \times 2 \times 1$. In other words, you are using the Fundamental Counting Principle to multiply the options for each "unfilled position" in your order which starts at n and decreases by 1 every time a position in the order is filled.

Factorial notation is used to write this operation when we do math:

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

$n!$ is read as "n-factorial".

You can only use non-negative whole numbers as n in a factorial and there is a special case: $0! = 1$.

Examples:

1. $1! =$
2. $5! =$
3. $11! =$

Exercise: Password

James and Sirius are trying to figure out the password for a secret passage behind the mirror on the fourth floor. They know the 7-digit long password uses the whole numbers from 1 to 7 but they do not know what order.

How many different passwords could there be if the digits **can repeat**?

How many different passwords could there be if the digits **cannot repeat**?

Permutations

What if we had n objects in total to choose from but we only needed to order k of them? **Permutations** are a way of counting in this type of situation where **order matters** and there is **no repetition**.

$$_nP_k = \frac{n!}{(n-k)!}$$

This is read as "n permute k" and counts how many ways we can order k objects from a total of n objects.

Exercise: Quidditch Tryouts

Bulgaria's National Quidditch team is holding tryouts for one Keeper, one Seeker, and one Chaser. Dimitar, Viktor, Georgi, Boris, Bogomil, Nikola, and Stoyanka all try out.

How many players total do we have to choose from and how many do we need to choose?

Does order matter when we choose our players? Why?

Is repetition allowed in our choices? Why?

Use the Fundamental Counting Principle to find how many ways the three positions can be filled.

Now use the permutation formula to find how many ways the three positions can be filled.

Combinations

What if we had n objects in total and needed to choose k with **no repetition** (just like in a permutation) but now **order does not matter**? This is called a **combination**.

$$_nC_k = \frac{n!}{k!(n-k)!}$$

This is read as "n choose k" and counts how many ways we can choose k objects from a total of n objects.

Exercise: Chocolate Frog Cards
Theodore Nott loses a bet to Blaise Zabini and has to give Blaise 4 of his chocolate frog cards. If Theodore has 49 cards in his collection, how many ways can Blaise pick which cards he'll take?

How many cards total do we have to choose from and how many do we need to choose?

Does order matter when we choose our players? Why?

Is repetition allowed in our choices? Why?

Now use the combination formula to find how many ways Blaise can pick the cards.

Bonus Exercise: Seating Order

Harry, Ron, Hermione, Moody, Tonks, Lupin, Teddy and Hagrid are going to watch a Quidditch game together. They find a free bench so they can all sit in a line. If Tonks, Lupin, and Teddy want to sit beside each other, then how many ways can the group be seated?

Problem Set

1. Seamus is trying to brew a potion but he's forgotten his textbook and doesn't remember the ingredients. He knows that he needs 1 plant, 1 animal part, and 1 powder. The ingredients available in class for each category are:

   - **Plant**: Mandrake root, Baneberry, Moonseed, or Wiggentree bark
   - **Animal Part**: Jobberknoll feather or Boomslang skin
   - **Powder**: Asphodel powder, Octopus powder, or Wartcap powder

   How many different potions would Seamus be able to make?

2. Gringotts vaults have a 4-digit numerical password. How many password combinations are there?

3. The Ministry of Magic's Department of Magical Transportation is trying out a new system where each fireplace in the Floo network has a Floo code much like the Canadian postal code. A Floo code is made up of 6 characters of the format "A1A 1A1" where A is a letter and 1 is a digit (from 0 to 9). How many different Floo codes could there be?

4. Solve these factorial problems:

   (a) $8! =$

   (b) $0! =$

   (c) $\dfrac{6!}{3!} =$

   (d) $\dfrac{17!}{14!3!2!} =$

5. Write these expressions using factorials:

   (a) $1 =$

   (b) $6 \times 5 \times 4 \times 3 \times 2 \times 1 =$

   (c) $100 \times 99 \times 98 \times 97 =$

   (d) $15 \times 14 \times 3 \times 2 \times 1 =$

   (e) $8 \times 7 \times 5 \times 3 \times 2 \times 1 =$

6. Professor McGonagall has to schedule the Quidditch matches between the 4 Hogwarts houses. Each house plays each of the other houses once.

   (a) How many matches in total are there?

   (b) How many different ways are there to schedule the matches?

7. Fred, George, Lee, Ginny, Bill, Ron, Dean, Seamus, and Charlie have a small, a medium, and a large chocolate frog. They have a broomstick race to decide who gets each of the frogs. The first, second, and third place winners will get the large, medium, and small frog respectively. How many ways can the frogs be given out?

8. Florean Fortescue's Ice Cream Parlour has 53 flavours of ice cream. How many different types of sundaes could Harry get if he wanted:

   (a) 1 scoop?

   (b) 2 differently flavoured scoops?

   (c) 5 differently flavoured scoops?

9. Bathilda Bagshot bought a bag of Bertie Bott's Beans. There are 21 bad tasting beans and 8 good tasting beans (all the beans are different). Bathilda wants to eat 4 beans.

   (a) How many ways can Bathilda pick her beans?

   (b) How many ways can Bathilda pick all good beans?

   (c) How many ways can Bathilda pick all bad beans?

   (d) How many ways can Bathilda pick 2 good beans AND 2 bad beans?

   (e) How many ways can Bathilda have more good beans than bad?

10. **Challenge problem.** Neville is reorganizing the greenhouse. How many ways can he rearrange Shrivelfig, Bubotuber, Devil's Snare, Gillyweed, Leaping Toadstools, Venomous Tentacula, Mandrake and Wolfsbane if he needs to keep the Mandrake and Wolfsbane plants together and Devil's Snare cannot be beside Gillyweed? Assume the greenhouse is thin and the plants go in a line.

11. **Challenge problem.** How many "words" can you make by shuffling this word:

    STATISTICS

    Note: These do not have to be real words but does have to be made up of the above 10 letters shuffled with no spaces or other symbols in between.

    Hint: Think about what is different between the formulas for permutations and combinations.

12. **Challenge problem.** Prove the following equations for any positive whole numbers n and k where $k \leq n$ by showing that the left and side and the right hand side of the equal sign can be written the same way.

    (a) ${}_nP_1 = {}_nC_1$

    (b) ${}_nC_k = {}_nC_{n-k}$

    (c) ${}_nP_0 = {}_nC_0 = 1$