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Grade 6 Math Circles

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Counting - Solutions

Introduction

You might be scratching your head right now wondering why our topic today is **counting**! Surely everyone here already knows how to count right? Are we going to spend the entire 2 hours of Math Circles counting from 1 to 1,000,000,000?!

What we will learn today is how to count the number of different possible outcomes of specific actions and events. This type of mathematical counting is the foundation of a entire branch of math called **combinatorics**. How is this different from counting 1, 2, 3,...? Well, we will learn mathematical "shortcuts" to counting up to very large numbers quickly and accurately!

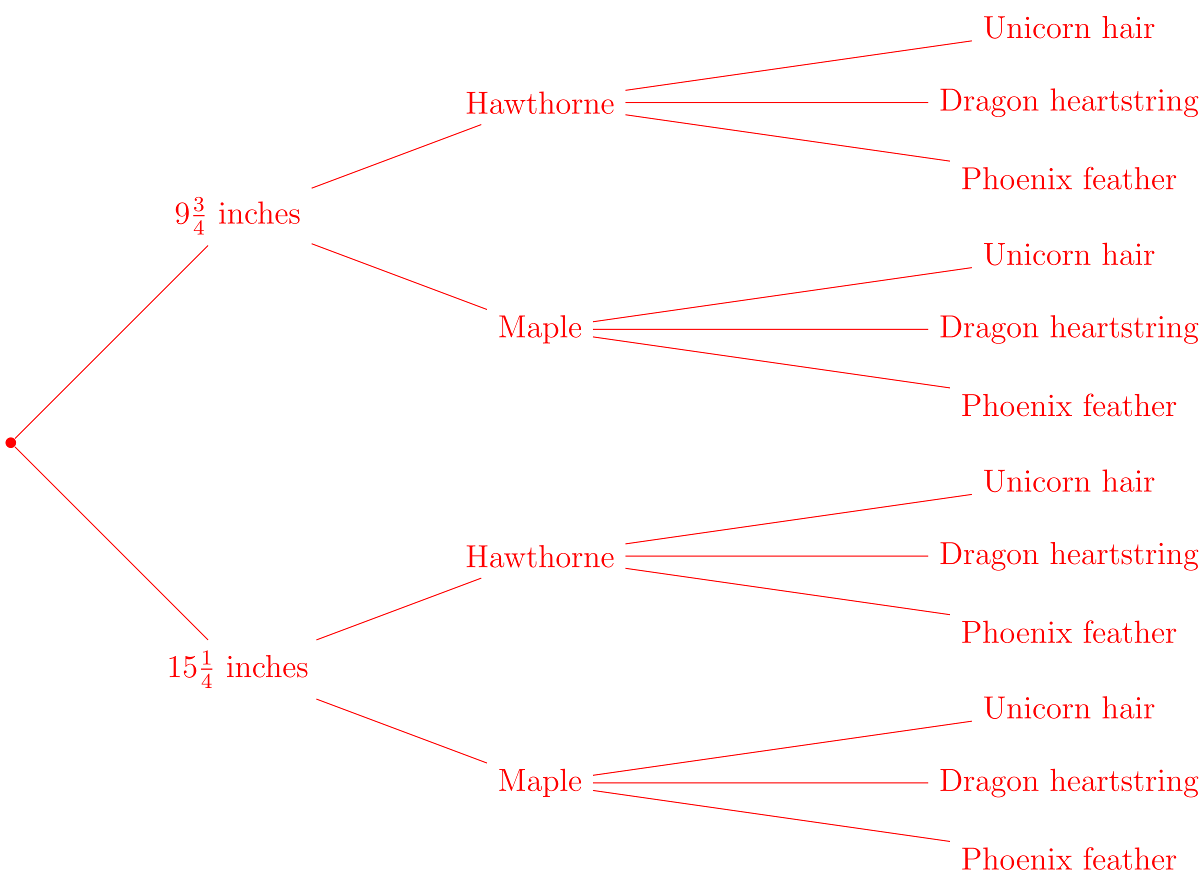
How many choices?

Exercise: Ollivander's Wand Making

Ollivander is making a new wand. These are the choices that he can make for the wand's length, wood, and core:

- **Length**: $9\frac{3}{4}$ inches or $15\frac{1}{4}$ inches
- **Wood**: Hawthorne or Maple
- **Core**: Unicorn hair, Dragon heartstring, or Phoenix feather

How many different types of wands can Ollivander make? Draw a tree diagram to show your work on the next page.



Thus, there are 12 different types of wands Ollivander can make.

Fundamental Counting Principle

If you have to make Choice A **AND** Choice B, and there are m options for Choice A and n options for Choice B, then the total number of different ways you can make Choice A and Choice B is $m \times n$.

When you see "**AND**" in a counting problem, it is a hint that you have to use this rule! Let's apply this rule to the Ollivander's Wand Making problem.

Exercise: Ollivander's Wand Making continued

How many choices do we have to make?

We have to choose a wand's length, wood, **AND** core so there are 3 choices.

How many options are there for each choice?

Length has 2 options, wood has 2 options, and core has 3 options.

Use the Fundamental Counting Principle to find the total number of different wands that Ollivander can make. Check that this is the same answer you got with the tree diagram before.

$2 \times 2 \times 3 = 12$.

Exercise: Luna's Outfit

Luna Lovegood is picking her outfit for the day. She needs to pick a top, bottom, accessory, and shoes. Here's what Luna has in her wardrobe:

- **Top**: Nargle infested sweater or Ravenclaw sweater (2 options)
- **Bottom**: Paint splattered skirt, blue wool leggings, or red shorts (3 options)
- **Accessory**: Butterbeer cork necklace, plum earrings, Spectrespecs, lion hat, eagle hat, or Ravenclaw scarf (6 options)
- **Shoes**: Loafers, sneakers, or high heels (3 options)

How many choices does Luna have to make?

4

How many different outfits could Luna put together?

$2 \times 3 \times 6 \times 3 = 108$ different outfits! Imagine having to draw a tree diagram for this!

How many orders?

Exercise: Potions Problem

Professor Snape gives Harry, Hermione, Draco, and Pansy an extra essay assignment as punishment for fighting in Potions class! Each student could get one of 6 grades: Outstanding, Exceeds Expectations, Acceptable, Poor, Dreadful, or Troll. Snape will then write the grades on the board because he's just that mean!

Is repetition allowed in the outcome (the grades that Snape writes on the board)?

Yes, multiple people could receive the same grade so the same grade can appear on the board more than once.

How many different outcomes are there? (i.e. How many different ways can these students be graded?) Use Fundamental Counting Principle.

$$\begin{aligned}\text{Number of ways} &= (\text{options for Harry}) \times (\text{options for Hermione}) \times (\text{options for Draco}) \times (\text{options for Pansy})\\ &= 6 \times 6 \times 6 \times 6\\ &= 6^4\\ &= 1296\end{aligned}$$

Exercise: Professors Wanted

Dumbledore is interviewing four candidates: Galatea Merrythought, Septima Vector, Vindictus Viridian, and Herbert Beery. All four of these professors are qualified to teach Potions, Arithmancy, Herbology, and Defense Against the Dark Arts (DADA) but each can only be assigned to one subject.

Is repetition allowed?
No, the same teacher cannot be chosen for more than one class.

How many different ways can Dumbledore match up the professors with the classes?

We know that Dumbledore needs to choose Professors for Potions, Arithmancy, Herbology, **AND** DADA. Therefore, we know that we need to use the Fundemental Counting Principle (ie. multipy the options). We need to calculate:

(options for Potions)×(options for Arithmancy)×(options for Herbology)×(options for DADA)

There are 4 options for Potions (since all 4 candidates are still unassigned) so our equation becomes:

$$4 \times (\text{options for Arithmancy}) \times (\text{options for Herbology}) \times (\text{options for DADA})$$

1 of the candidates has now been assigned so for Arithmancy, there are 3 options left. So, our equation becomes:

$$4 \times 3 \times (\text{options for Herbology}) \times (\text{options for DADA})$$

After that, two candidates will have already been assigned so there are 2 options left for Arithmancy and after that, only 1 professor is left unassigned for DADA. Thus, our equation is:

$$4 \times 3 \times 2 \times 1 = 24$$

Factorials

Whenever we have a list or group of n things and we need to figure out how many different orders they can be put in (how many different shuffles) with **no repetition**, we use the same type of logic that we just applied to the previous problem. This will always result in the answer being $n \times (n-1) \times (n-2) \times (n-3) \times ... \times 2 \times 1$. In other words, you are using the Fundamental Counting Principle to multiply the options for each "unfilled position" in your order which starts at n and decreases by 1 every time a position in the order is filled.

Factorial notation is used to write this operation when we do math:

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

$n!$ is read as "n-factorial".

You can only use non-negative whole numbers as n in a factorial and there is a special case: $0! = 1$.

Examples:

1. $1! = 1$
2. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
3. $11! = 11 \times 10 \times ... \times 3 \times 2 \times 1 = 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5! = 39,916,800$

Exercise: Password

James and Sirius are trying to figure out the password for a secret passage behind the mirror on the fourth floor. They know the 7-digit long password uses the whole numbers from 1 to 7 but they do not know what order.

How many different passwords could there be if the digits **can repeat**?

There are 7 options for each digit, so:

$7 \times 7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^7 = 823,543$

How many different passwords could there be if the digits **cannot repeat**?

Because there are no repeats, every time you fill a digit the number of options for the next digit decreases by one, so:

$7! = 5040$

Permutations

What if we had n objects in total to choose from but we only needed to order k of them?

Permutations are a way of counting in this type of situation where **order matters** and there is **no repetition**.

$$_nP_k = \frac{n!}{(n-k)!}$$

This is read as "n permute k" and counts how many ways we can order k objects from a total of n objects.

Exercise: Quidditch Tryouts

Bulgaria's National Quidditch team is holding tryouts for one Keeper, one Seeker, and one Chaser. Dimitar, Viktor, Georgi, Boris, Bogomil, Nikola, and Stoyanka all try out.

How many players total do we have to choose from and how many do we need to choose?
There are 7 total players to choose from total and we need to choose 3.

Does order matter when we choose our players? Why?
Yes, order does matter because we need to assign each player to a position. If we need to assign (Keeper, Seeker, Chaser), then choosing (Viktor, Georgi, Boris) is different than (Gorgi, Boris, Victor).

Is repetition allowed in our choices? Why?
No, there is no repetition allowed because once a player is picked for one position, they can't also be picked for another.

Use the Fundamental Counting Principle to find how many ways the three positions can be filled.

$$\begin{aligned}
\text{Number of ways} &= (\text{options for Keeper}) \times (\text{options for Seeker}) \times (\text{options for Chaser}) \\
&= 7 \times 6 \times 5 \\
&= \frac{7!}{4 \times 3 \times 2 \times 1} \\
&= \frac{7!}{4!} \\
&= 210
\end{aligned}$$

Now use the permutation formula to find how many ways the three positions can be filled.
We already figured out that $n = 7$ and $k = 3$.

$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4!}{4!} = 7 \times 6 \times 5 = 210$$

Combinations

What if we had n objects in total and needed to choose k with **no repetition** (just like in a permutation) but now **order does not matter**? This is called a **combination**.

$$_nC_k = \frac{n!}{k!(n-k)!}$$

This is read as "n choose k" and counts how many ways we can choose k objects from a total of n objects.

Exercise: Chocolate Frog Cards
Theodore Nott loses a bet to Blaise Zabini and has to give Blaise 4 of his chocolate frog cards. If Theodore has 49 cards in his collection, how many ways can Blaise pick which cards he'll take?

How many cards total do we have to choose from and how many do we need to choose?
There are 49 total cards to choose from total and we need to choose 4.

Does order matter when we choose our players? Why?
No, order does not matter because Blaise ends up with the same 4 cards no matter which he picks first, second, and third.

Is repetition allowed in our choices? Why?
No, there is no repetition allowed because once a card is picked, it can't be picked a second time.

Now use the combination formula to find how many ways Blaise can pick the cards.

We already figured out that $n = 49$ and $k = 4$.

$$_{29}C_4 = \frac{49!}{4!(49-4)!} = \frac{49!}{4!45!} = \frac{49 \times 48 \times 47 \times 46}{4!} = 211,876$$

Bonus Exercise: Seating Order

Harry, Ron, Hermione, Moody, Tonks, Lupin, Teddy and Hagrid are going to watch a Quidditch game together. They find a free bench so they can all sit in a line. If Tonks, Lupin, and Teddy want to sit beside each other, then how many ways can the group be seated?

Order matters and there is no repetition so we use a factorial. The trick is that, because Tonks, Lupin, and Teddy have to sit next to each other, think of them as one person. The number of ways to arrange Harry, Ron, Hermione, Moody, Tonks/Lupin/Teddy, and Hagrid is $6! = 720$ since they count as 6 people.

But, we also have to consider how Tonks, Lupin, and Teddy are sitting. There are 720 ways the group can be arranged for each arrangement Tonks, Lupin, and Teddy can make. We are now looking at how many ways to order Tonks, Lupin, and Teddy which is $3! = 6$.

So, the total number of ways the group can sit is: $6! \times 3! = 720 \times 6 = 4320$

Problem Set

1. Seamus is trying to brew a potion but he's forgotten his textbook and doesn't remember the ingredients. He knows that he needs 1 plant, 1 animal part, and 1 powder. The ingredients available in class for each category are:

   - **Plant**: Mandrake root, Baneberry, Moonseed, or Wiggentree bark
   - **Animal Part**: Jobberknoll feather or Boomslang skin
   - **Powder**: Asphodel powder, Octopus powder, or Wartcap powder

   How many different potions would Seamus be able to make?

   From Fundamental Counting Principle: $4 \times 2 \times 3 = 24$.

2. Gringotts vaults have a 4-digit numerical password. How many password combinations are there?

   Since digits can repeat and there are 10 options for each digit (0 to 9), by the Fundamental Counting Principle: $10^4 = 10,000$

3. The Ministry of Magic's Department of Magical Transportation is trying out a new system where each fireplace in the Floo network has a Floo code much like the Canadian postal code. A Floo code is made up of 6 characters of the format "A1A 1A1" where A is a letter and 1 is a digit (from 0 to 9). How many different Floo codes could there be?

   There are 26 letters in the alphabet and 10 digits (from 0 to 9). Repetition is also allowed because the letters or numbers can repeat. So, according to Fundamental Counting Principle:

   $$26 \times 10 \times 26 \times 10 \times 26 \times 10 = 26^3 \times 10^3 = 17,576,000$$

4. Solve these factorial problems:

   (a) $8! = 40,320$

   (b) $0! = 1$

   (c) $\frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3!}{3!} = 6 \times 5 \times 4 = 120$

   (d) $\frac{17!}{14!3!2!} = \frac{17 \times 16 \times 15}{(3 \times 2 \times 1) \times (2 \times 1)} = \frac{17 \times 16 \times 15}{3 \times 4} = 17 \times 4 \times 5 = 340$

5. Write these expressions using factorials:

   (a) $1 = 1! = 0! = \frac{n!}{n!}$

   (b) $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$

   (c) $100 \times 99 \times 98 \times 97 = \dfrac{100!}{96!}$

   (d) $15 \times 14 \times 3 \times 2 \times 1 = \dfrac{15!}{13!} \times 3! = \dfrac{15!3!}{13!}$

   (e) $8 \times 7 \times 5 \times 3 \times 2 \times 1 = \dfrac{8!}{6 \times 4} = \dfrac{8!}{\frac{6!}{5!} \times \frac{4!}{3!}} = \dfrac{8!5!3!}{6!4!}$

6. Professor McGonagall has to schedule the Quidditch matches between the 4 Hogwarts houses. Each house plays each of the other houses once.

   (a) How many matches in total are there?

   $$3 + 2 + 1 = 6 \text{ matches.}$$

   Slytherin has plays the 3 other houses, then Hufflepuff plays the 3 other houses but it's already played Slytherin so that match is already counted. Then Ravenclaw plays the 3 other houses but it's already played Slytherin and Hufflepuff so it only has to play Gryffindor. And now, Gryffindor has already played all other houses.

   (b) How many different ways are there to schedule the matches?

   We need to arrange 6 games with no repeats and when order matters so:

   $$6! = 720$$

7. Fred, George, Lee, Ginny, Bill, Ron, Dean, Seamus, and Charlie have a small, a medium, and a large chocolate frog. They have a broomstick race to decide who gets each of the frogs. The first, second, and third place winners will get the large, medium, and small frog respectively. How many ways can the frogs be given out?

   Order matters because it determines who gets which frog. There is no repetition because someone cannot finish the race twice. So, we use permutation. There are 9 competitors in total and 3 chocolate frogs.

   $${}_9P_3 = \frac{9!}{(9-3)!} = \frac{9!}{6!} = 9 \times 8 \times 7 = 504$$

8. Florean Fortescue's Ice Cream Parlour has 53 flavours of ice cream. How many different types of sundaes could Harry get if he wanted:

   (a) 1 scoop?

   Logically, there are 53 different sundaes possible with 1 scoop. Mathematically, ${}_{53}C_1 = {}_{53}P_1 = 53$.

   (b) 2 differently flavoured scoops?

   Order does not matter and there is no repetition (since you want different scoops) so this is a combination problem.

$${}_{53}C_2 = \frac{53!}{2!(53-2)!} = \frac{53!}{2!51!} = 1378$$

   (c) 5 differently flavoured scoops?

$${}_{53}C_5 = \frac{53!}{5!(53-5)!} = \frac{53!}{5!48!} = 2,869,685$$

9. Bathilda Bagshot bought a bag of Bertie Bott's Beans. There are 21 bad tasting beans and 8 good tasting beans (all the beans are different). Bathilda wants to eat 4 beans.

   (a) How many ways can Bathilda pick her beans?

   Order does not matter and there is no repetition so this is a combination question. There are 21 + 8 = 29 beans total and she needs to pick 4.

$${}_{29}C_4 = \frac{29!}{4!(29-4)!} = \frac{29!}{4!25!} = 23,751$$

   (b) How many ways can Bathilda pick all good beans?

   There are 8 good beans in total and she needs to pick 4.

$${}_{8}C_4 = \frac{8!}{4!(8-4)!} = \frac{8!}{4!4!} = 70$$

   (c) How many ways can Bathilda pick all bad beans?

   There are 21 bad beans in total and she needs to pick 4.

$${}_{21}C_4 = \frac{21!}{4!(21-4)!} = \frac{21!}{4!17!} = 5985$$

   (d) How many ways can Bathilda pick 2 good beans AND 2 bad beans?

   The "and" tells us that we need to use the Fundamental Counting Principle in this question. We need to multiply the number of ways Bathilda can pick 2 good beans with the number of ways Bathilda can pick 2 bad beans.

$$_8C_2 \times \ _{21}C_2 = \frac{8!}{2!(8-2)!} \times \frac{21!}{2!(21-2)!} = \frac{8!}{2!(6)!} \times \frac{21!}{2!(19)!} = 28 \times 210 = 5880$$

(e) How many ways can Bathilda have more good beans than bad?

There are only two scenarios in which Bathilda has more good than bad beans: when she has 3 good and 1 bad or when she has all good beans. Thus, we have to add the number of ways for each situation. Following the same logic as the previous parts:

$$\begin{aligned}
(\text{3 good and 1 bad}) + (\text{all good}) &= (_8C_3 \times \ _{21}C_1) + \ _8C_4 \\
&= (\frac{8!}{3!(8-3)!} \times \frac{21!}{1!(21-1)!}) + 70 \\
&= (\frac{8!}{3!(85)!} \times \frac{21!}{1!(20)!}) + 70 \\
&= (56 \times 21) + 70 \\
&= 1246
\end{aligned}$$

10. **Challenge problem.** Neville is reorganizing the greenhouse. How many ways can he rearrange Shrivelfig, Bubotuber, Devil's Snare, Gillyweed, Leaping Toadstools, Venomous Tentacula, Mandrake and Wolfsbane if he needs to keep the Mandrake and Wolfsbane plants together and Devil's Snare cannot be beside Gillyweed? Assume the greenhouse is thin and the plants go in a line.

This is similar to the bonus problem. Because Mandrake and Wolfsbane need to stay together, we need to think of them as one plant and then multiply by the number of ways those two plants can be arranged. If we think of Mandrake and Wolfsbane as 1 plant, then we count 6 plants in total to arrange. So:

$$6! \times 2! = 1440$$

We also have the restriction that Devil's Snare cannot be beside Gillyweed. Because of this, we need to subtract the number of ways Devil's Snare **can** be beside Gillyweed (while Mandrake and Wolfsbane are also beside each other). If we think of Mandrake and Wolfsbane as 1 plant and also Devil's Snare and Gillyweed as one plant, then we count 5 plants. Thus the number of ways both Mandrake and Wolfsbane are together **AND** Devil's Snare and Gillyweed are together is:

$$5! \times 2! \times 2! = 480$$

Thus, our final answer is:

$$(6! \times 2!) - (5! \times 2! \times 2!) = 1440 - 480 = 960$$

11. **Challenge problem.** How many "words" can you make by shuffling this word:

STATISTICS

Note: These do not have to be real words but does have to be made up of the above 10 letters shuffled with no spaces or other symbols in between.

Hint: Think about what is different between the formulas for permutations and combinations.

There are 10 letters to shuffle but the letters have repeats:

- 3 of S
- 3 of T
- 2 of I

To deal with these repeats, we divide by the number of ways these repeated letters can be arranged because they do not result in "different" words. (ie. if you shuffle the word "AAB", you will only have $\frac{3!}{2!} = 3$ unique words which are "BAA", "ABA", and "AAB" rather than $3! = 6$ because it does not matter how you arrange the 2 A's.)

So, our answer is:

$\frac{10!}{3!3!2!} = 50,400$

12. **Challenge problem.** Prove the following equations for any positive whole numbers n and k where $k \leq n$ by showing that the left and side and the right hand side of the equal sign can be written the same way.

(a) ${}_nP_1 = {}_nC_1$

$$\text{Left hand side} = {}_nP_1$$
$$LHS = \frac{n!}{(n-1)!}$$

$$\begin{aligned}\text{Right hand side} &= {}_nC_1 \\ RHS &= \frac{n!}{1!(n-1)!} \\ &= \frac{n!}{(n-1)!}\end{aligned}$$

Thus, LHS = RHS so we have proved ${}_nP_1 = {}_nC_1$. □

(b) ${}_nC_k = {}_nC_{n-k}$

$$\begin{aligned}\text{Left hand side} &= {}_nC_k \\ LHS &= \frac{n!}{k!(n-k)!}\end{aligned}$$

$$\begin{aligned}\text{Right hand side} &= {}_nC_{n-k} \\ RHS &= \frac{n!}{(n-k)!(n-(n-k))!} \\ &= \frac{n!}{(n-k)!(n-n+k)!} \\ &= \frac{n!}{(n-k)!(k)!} \\ &= \frac{n!}{k!(n-k)!}\end{aligned}$$

Thus, LHS = RHS so we have proved ${}_nC_k = {}_nC_{n-k}$. □

(c) ${}_nP_0 = {}_nC_0 = 1$

$$\begin{aligned}\text{Left hand side} &= {}_nP_0 \\ LHS &= \frac{n!}{(n-0)!} \\ &= \frac{n!}{n!} \\ &= 1\end{aligned}$$

$$\begin{aligned}
\text{Right hand side} &= {}_nC_0 \\
RHS &= \frac{n!}{0!(n-0)!} \\
&= \frac{n!}{1(n!)} \\
&= 1
\end{aligned}$$

Thus, LHS = RHS = 1 so we have proved ${}_nP_0 = {}_nC_0 = 1$. $\square$