UNIVERSITY OF
WATERLOO
Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

Winter 2019—March 5 & 6

Platonic Solids

Tessellations

"Geometry is the archetype of beauty in the world."—Johannes Kepler, Astronomer

As you've already seen this term, it is possible (and often very pretty) to arrange certain shapes across a flat surface that extends as far as you can imagine. You probably noticed, however, that not every shape forms a nice sheet with no gaps. Why is that? It comes down to whether the shapes have the appropriate angles. If a shape has interior angles that are strange, they don't add up nicely and therefore cannot tile in a sheet of paper.

What if we wanted to *extend* the idea of a tessellation? When we extend something in mathematics, it usually means that we want to take an existing idea and make it less specific so we can apply it to more things. In this case, what details are we going to leave behind? We are going to leave behind two dimensions—rather than working on a sheet of paper, we'll work in 3D space. Additionally, we're not going to worry about whether our shapes fill in a sheet of paper (or a given box) with no gaps. We're only interested in how to fit shapes around a corner.

2D to 3D

In two dimensions, you can create as many regular polygons (shapes with sides that are all the same length and angles that are all the same) as you like. In 3D space, however, there are only a few shapes that we can arrange around a corner in order to form a nice solid shape. The 2D shapes that we can use to make nice, regular 3D objects are as follows: triangles, squares, and pentagons. We're going to see exactly how to do it.

Constructing the platonic solids

We'll go in order of least to most sides, beginning with the tetrahedron. Fold the three outside triangles into the middle so they meet and tape them together.

For the cube, fold the squares that are immediately beside the middle square up. You should have what looks like a box with a lid sticking up. Fold down the "lid" and tape as necessary.

For the octahedron, picture the net as two Pac-mans (Pac-men?) stuck together. Fold the right Pac-man into a pyramid by folding its mouth shut. Do the same on the left and tape as necessary.

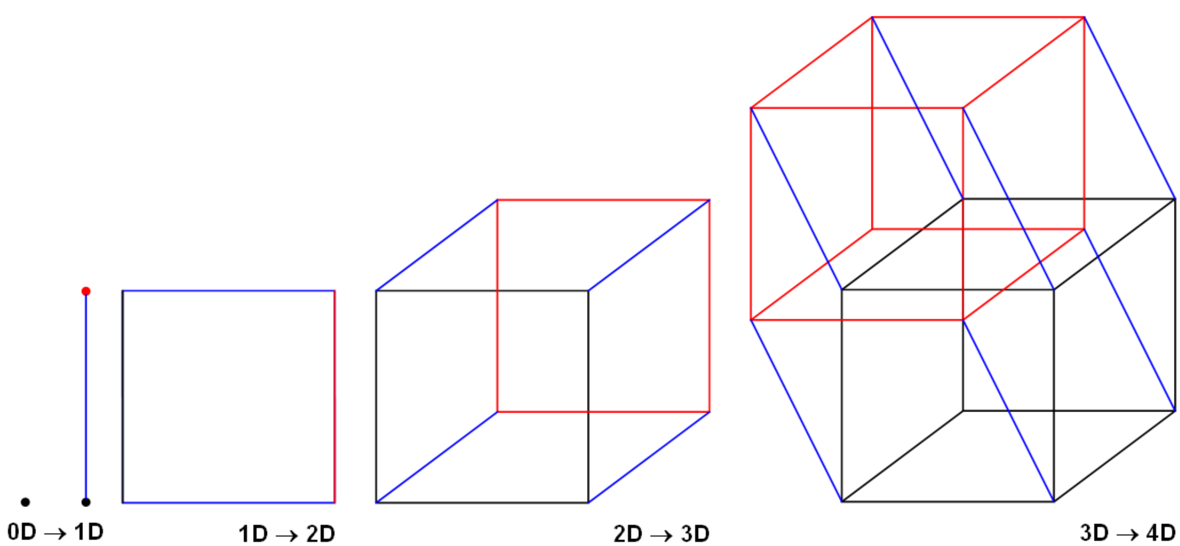
For the dodecahedron, fold the top pentagons towards the middle until their edges meet and tape. Do the same for the bottom. Then join the two halves and tape.

Finally, for the icosahedron, fold the main diagonal into a ring and tape closed. Then fold the triangular "teeth" along the top and bottom inwards and tape.

We call the shapes that we just made the *platonic solids* after Plato. He might not have been the first to study them, but he was the most interested in them as far as history can tell. They are special because they are the only 3D objects that have all faces that are the same size, shape, and every face at the same angle to every other. If you think about it, that's pretty amazing. Out of the infinite number of 3D shapes that you can possibly imagine there are only five that are this nice! That is why Plato and others were so interested in them.

3D and beyond

Now suppose we want to extend the idea of platonic solids to even more dimensions. "Alright," you're probably thinking, "Math circles has gone crazy." Actually, thanks to math, we can extend the idea of spacial dimensions themselves. To imagine this, let's look at the following diagram.



What is happening here? The first movement, 0D to 1D, makes sense. You may not be used to thinking like this, but if you think of "dimension" as "number of directions you can move," it makes sense to think of a point as being zero-dimensional—you can't move at all! After that, 1D lets you move up and down. Now suppose you wanted to go left and right. What would you do? Extend your line sideways into a plane. You can do the same thing to go to 3D. If you want to move forwards and backwards, take your plane and move perpendicularly (at a right angle) away from itself to get a cube. Keep in mind that so far, we've connected each corner with each other. This is where it gets tricky. In order to imagine four or more dimensions, we have to accept the idea of there being even more perpendicular directions to move. There aren't many good ways to get a feel for this without some sort of technology to help us. There are a variety of useful games including 4D Rubik's cube simulators and 4D "Tetris"-es. All of these are mind-bending but understandable with practice.

To extend the idea of platonic solids even further, try imagining putting our 3D platonic solids around a central point that we can fold into a 4D-corner. It's the same idea as with the 3D platonic solids, just that the sides or faces are now what we call "cells." Just as in 3D, only certain shapes fold nicely into shapes with no gaps. It turns out that in 4D, there are actually six platonic solids, and in dimensions higher than 4, there are only 3. Why is this? In 4D, there are generalizations of all of the platonic solids we discovered today, plus a bonus one. This is because of how the angles work out specifically in 4D. However, after 4D, we "run out of room" in a sense. Angles become too crowded and we can only find generalizations of the tetrahedron, octohedron, and cube. We call these the "simplex," "cross-polytope," and "hypercube." (A polytope is just a shape in a number of dimensions greater than three.)

Why?

You may ask, "what use do we have for higher-dimensional shapes?" It turns out that the idea of more spacial dimensions is incredibly useful in the areas of statistics and physics. In statistics, we can model the outcomes of a group of many experiments as ordered tuples. A tuple is the generalized idea of a pair—a collection whose individual items "go together," so to speak. If we do this, we can think of the tuples as points in high-dimensional space. We can then do geometry with these points, allowing us to draw conclusions we otherwise could not. In physics, there are a variety of theories that require higher-dimensional geometry to make sense. Einstein's theories of relativity both require that we treat space and time as one interwoven, four-dimensional reality. Additionally, the modern string theories typically make use of 11 spacial dimensions to be mathematically consistent.

Thinking in higher dimensions is extraordinarily difficult, but somewhat doable thanks to modern technology. Good luck with the following problems!

Problems

"*" denotes a challenge problem

1. With your parents' permission, head on over to http://superliminal.com/cube/cube.htm and give any of their puzzles a try. They are quite difficult but give an idea of how rotations in 4D work. There are also several other puzzles for your solving pleasure.

2. Which of the platonic solids can be used to tessellate (fill in) 3D space?

3. Why aren't there more platonic solids? To find out, arrange as many regular triangles, squares, and pentagons as you can around a central point without making a nice tessellation. What common feature is there among the sums of interior angles of the polygons? Now try fitting three hexagons around a common point. What is the sum of their interior angles? Draw a conclusion about the condition for a platonic solid to exist whose face is a given regular polygon.

4. * What is the relationship between the number of vertices, cells, and number of spacial dimensions for the...

   (a) ...simplex,

   (b) ...cross-polytope,

   (c) ...and hypercube

   Hint: begin by checking 3 and 4D and see if you can figure out the pattern.