Problem Set 2 Solutions

Intermediate Math Circles Fall 2019
More Fun With Inequalities

Rational Inequalities

1. $\frac{1}{x} \leq -7, x \neq 0$

   Solution:

   Case 1 $[x > 0]$

   $\frac{1}{x} \leq -7$

   $1 \leq -7x$

   $\frac{1}{-7} \geq \frac{-7x}{-7}$

   $x \leq \frac{-1}{7}$

   But $x > 0$ and $x \leq \frac{-1}{7}$ isn't possible.

   Thus this case is impossible.

   Case 2 $[x < 0]$

   $\frac{1}{x} \leq -7$

   $1 \geq -7x$

   $\frac{1}{-7} \leq \frac{-7x}{-7}$

   $x \geq \frac{-1}{7}$

   Thus $\frac{-1}{7} \leq x < 0$

2. $\frac{3}{x-2} \geq \frac{1}{4}, x \neq 2$

   Solution:

Case 1 $[(x-2)>0]$

$\frac{3}{x-2} \geq \frac{1}{4}$

$12 \geq x-2$

$12+2 \geq x$

$x \leq 14$

If $x-2>0$, we know $x>2$ as well as $x \leq 14$

Thus, $2<x \leq 14$

Case 2 $[(x-2)<0]$

$\frac{3}{x-2} \geq \frac{1}{4}$

$12 \leq x-2$

$12+2 \leq x$

$x \geq 14$

If $x-2<0$, then $x<2$. But there is no x such that $x<2$ and $x \geq 14$.

This case is impossible

3. $\frac{x-3}{x+1}<2, x \neq -1$

Solution:

Case 1 $[(x+1)>0]$

$\frac{x-3}{x+1}<2$

$x-3<2(x+1)$

$x-3<2x+2$

$-3-2<2x-x$

$-5<x$

If $x+1>0$, then $x>-1$ must also hold. Luckily if $x>-1$, then $x>-5$.

Thus, for this case $x>-1$

Case 2 $[(x+1)<0]$

$\frac{x-3}{x+1}<2$

$x-3>2(x+1)$

$x-3>2x+2$

$-3-2>2x-x$

$-5>x$

If $x+1<0$, then $x<-1$ must also hold. Luckily if $x<-5$, then $x<-1$.

Thus, for this case $x<-5$

Therefore, when $x > -1$ or when $x < -5$ the inequality is satisfied.

Absolute Values

Solve each of the following algebraically. Check your answer graphically.

(a) $|x + 6| = 5$

Solution: Case 1:$[(x + 6) \geq 0]$

$$\begin{aligned} |x+6| &= 5 \\ (x+6) &= 5 \\ x &= 5 - 6 \\ x &= -1 \end{aligned}$$

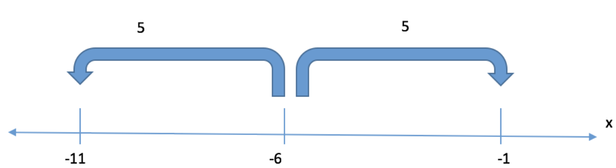
Case 2: $[(x + 6) < 0]$

$$\begin{aligned} |x+6| &= 5 \\ -(x+6) &= 5 \\ -x - 6 &= 5 \\ -x &= 11 \\ x &= -11 \end{aligned}$$

Therefore x = -1 and x = -11 satisfy the equation.

Check
Where is the special point?

$$\begin{aligned} x + 6 &= 0 \\ x &= -6 \end{aligned}$$



(b) $|x - 4| \geq 1$

Solution: Case 1: $[(x-4) \geq 0]$

$$\begin{aligned} |x-4| &\geq 1 \\ (x-4) &\geq 1 \\ x-4 &\geq 1 \\ x &\geq 5 \end{aligned}$$

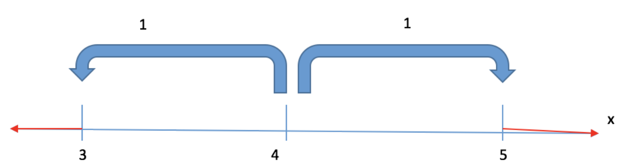
Case 2: $[(x-4) < 0]$

$$\begin{aligned} |x-4| &\geq 1 \\ -(x-4) &\geq 1 \\ -x+4 &\geq 1 \\ 4-1 &\geq x \\ 3 &\geq x \\ x &\leq 3 \end{aligned}$$

Therefore when $x \geq 5$ or when $x \leq 3$ the inequality is satisfied.

Check
Where is the "special" point?

$$\begin{aligned} x-4 &= 0 \\ x &= 4 \end{aligned}$$



(c) $|2x+1| < 7$

Solution: Case 1 [(2x+1)≥0]:

$$|2x+1| < 7$$
$$2x+1 < 7$$
$$2x < 7-1$$
$$\frac{2x}{2} < \frac{6}{2}$$
$$x < 3$$

Case 2 [(2x+1)<0]:

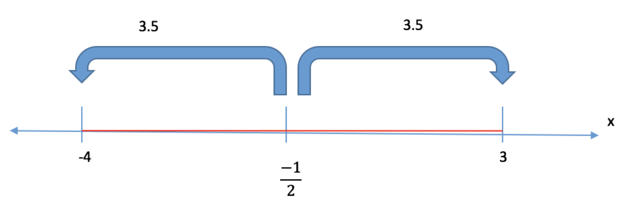
$$|2x+1| < 7$$
$$-(2x+1) < 7$$
$$-2x-1 < 7$$
$$-2x < 8$$
$$\frac{-2x}{-2} > \frac{8}{2}$$

We flip the inequality here!

$$x > -4$$

Therefore the inequality holds for -4<x<3.
Check
Where is our special point?

$$2x+1 = 0$$
$$x = \frac{-1}{2}$$



Wait! Why are we only going 3.5 from $\frac{-1}{2}$?
Because the x value is being doubled within this inequality.
In other words, we have 2x vs x.

(d) $|x-2| + |x+5| = 8$

Solution: Case 1 [(x-2) $\geq$ 0 and (x+5) $\geq$ 0]:

$$\begin{aligned} |x-2|+|x+5| &= 8 \\ (x-2)+(x+5) &= 8 \\ 2x+3 &= 8 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

Case 2 [(x-2) $\geq$ 0 and (x+5) $<$ 0]:

$$\begin{aligned} |x-2|+|x+5| &= 8 \\ x-2+[-(x+5)] &= 8 \\ x-2-x-5 &= 8 \\ -7 &= 8 \end{aligned}$$

Case is inadmissable.

Case 3 [(x-2) $<$ 0 and (x+5) $\geq$ 0]:

$$\begin{aligned} |x-2|+|x+5| &= 8 \\ -(x-2)+(x+5) &= 8 - x+2+x+5 \qquad = 8 \\ 7 &= 8 \end{aligned}$$

Case is inadmissable.

Case 4 [(x-2) $<$ 0 and (x+5) $<$ 0]:

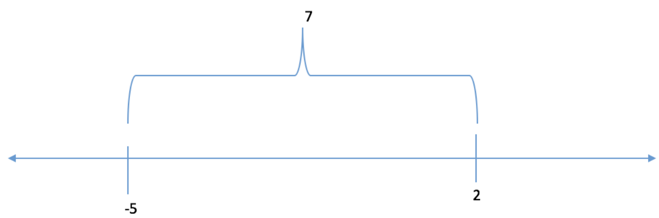
$$\begin{aligned} |x-2|+|x+5| &= 8 \\ -(x-2)+[-(x+5)] &= 8 \\ -x+2+(-x-5) &= 8 \\ -x+2-x-5 &= 8 \\ -2x-3 &= 8 \\ -2x &= 11 \\ x &= \frac{-11}{2} \end{aligned}$$

Therefore x = $\frac{5}{2}$ and x = $\frac{-11}{2}$ satisfy the equation.

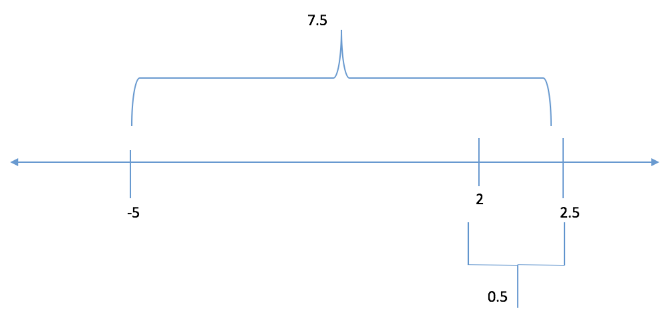
Check

Where are the special points?

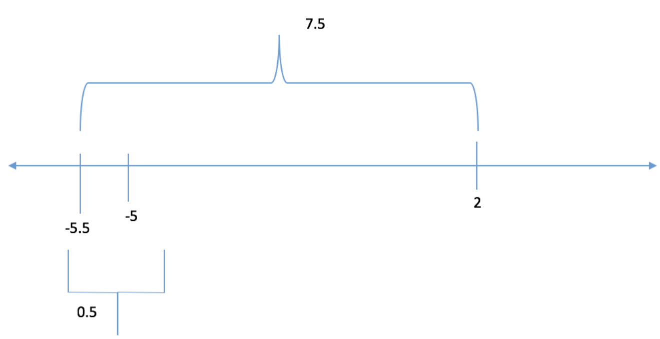
$$\begin{aligned} x - 2 &= 0 \\ x &= -2 \\ x + 5 &= 0 \\ x &= -5 \end{aligned}$$



We know that for any x in the range -5 $\leq$ x $\leq$ 2 that |x-2| + |x+5| = 7. To the right of 2 we need to add 1 so 7 + 1 = 8.



Similarly to the left of -5,



(e) $|x| + |2 - x| \leq 12$

Solution: Case 1 [x $\geq$ 0 and (2-x) $\geq$ 0]:

$$|x| + |2 - x| \leq 12$$
$$x + 2 - x \leq 12$$
$$2 \leq 12$$

While this is true, it doesn't help us find the values that work.

Case 2 [x $\geq$ 0 and (2-x) $<$ 0]:

$$|x| + |2 - x| \leq 12$$
$$x + [-(2 - x)] \leq 12$$
$$x + (-2 + x) \leq 12$$
$$2x - 2 \leq 12$$
$$2x \leq 14$$
$$x \leq 7$$

Case 3 [x $<$ 0 and (2-x) $\geq$ 0]:

$$|x| + |2 - x| \leq 12$$
$$-x + 2 - x \leq 12$$
$$-2x + 2 \leq 12$$
$$-2x \leq 10$$
$$\frac{-2x}{-2} \geq \frac{10}{-2}$$

We flip the inequality.

$$x \geq -5$$

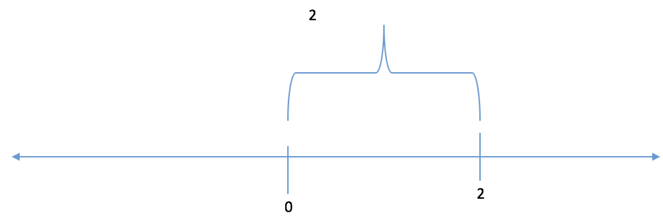
Case 4 [x $<$ 0 and (2-x) $<$ 0]:

$$|x| + |2 - x| \leq 12$$
$$-x + [-(2 - x)] \leq 12$$
$$-x - 2 + x \leq 12$$
$$-2 \leq 12$$

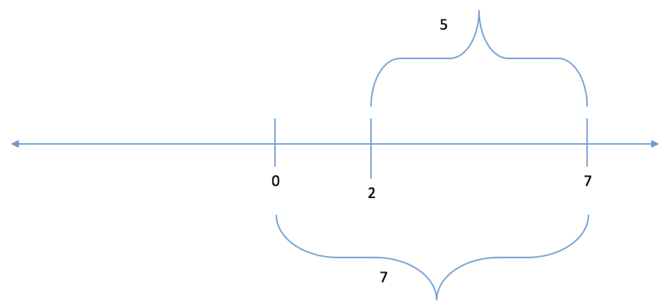
Again this is true, but doesn't help.
Therefore the inequality holds for -5 $\leq$ x $\leq$ 7.

Check
Where are the special points?

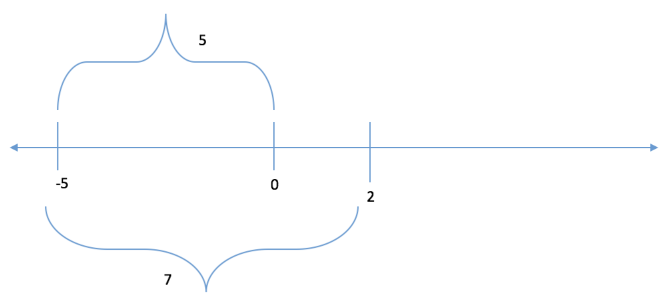
$$x = 0$$
$$2 - x = 0$$
$$2 = x$$
$$x = 2$$



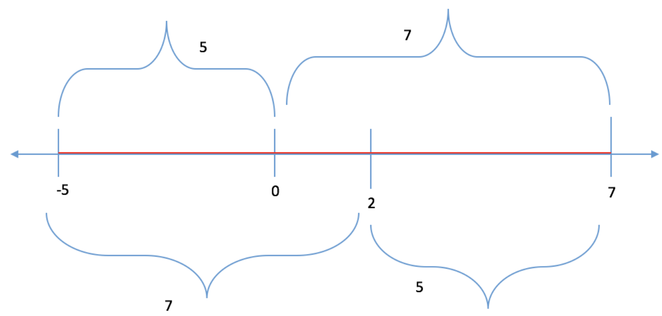
We know for any point in the range $0 < x < 2$, $|x| + |2-x| = 2$ and $2 \leq 12$. To the right of 2 we can add 10 so that $2 + 10 = 12$.



Similarly to the left of 0,



When we consider the last two number lines together, we get the following:



(f) $|x| \geq 7$

Solution:

Case 1 $[x \geq 0]$

$|x| \geq 7$

$x \geq 7$

Case 2 $[x < 0]$

$|x| \geq 7$

$-x \geq 7$

$x \leq -7$

Therefore, when $x \geq 7$ or when $x \leq -7$ the inequality is satisfied.

(g) $|x - 6| < 5$

Solution:

Case 1 $[(x-6) \geq 0]$

$|x-6| < 5$

$x-6 < 5$

$x < 11$

If $x-6 \geq 0$, then we know $x \geq 6$ as well as $x < 11$.

Thus, $6 \leq x < 11$.

Case 2 $[(x-6) < 0]$

$|x-6| < 5$

$-(x-6) < 5$

$-x+6 < 5$

$6-5 < x$

$x > 1$

If $x-6 < 0$, then we know $x < 6$ as well as $x > 1$.

Thus, $1 < x < 6$.

Therefore, when we combine the two inequalities we get $1 < x < 11$
.

(h) $|x+2| \geq 8$

Solution:

Case 1 $[(x+2) \geq 0]$

$|x+2| \geq 8$

$x+2 \geq 8$

$x \geq 6$

If $(x+2) \geq 0$, then we know $x \geq -2$. Luckily if $x \geq 6$, then $x \geq -2$ as well.

Thus, $x \geq 6$

Case 2 $[(x+2) < 0]$

$|x+2| \geq 8$

$-(x+2) \geq 8$

$-x-2 \geq 8$

$-2-8 \geq x$

$-10 \geq x$

$x \leq -10$

If $(x+2) < 0$, then we know $x < -2$. Again if $x \leq -1$, then $x < -2$ as well.

Thus, $x \leq -10$

Therefore, when $x \geq 6$ or when $x \leq -10$ the inequality is satisfied.

(i) $|3x| > 6$

Solution:
We know our "special" point is at zero with $|3x| > 6$ and we could use the argument we used in Problem Set 1 Question 4. That is, we need points $\frac{1}{3}$ of 6 away from our special point zero because we have $3x$ instead of x. Or we could use the fact that $|ab| = |a||b|$ discussed in Problem Set 1 Question 3 to show:

$|3x| > 6$

$|3||x| > 6$

$3|x| > 6$

$|x| > 2$

Therefore when $x < -2$ or $x > 2$ is the inequality satisfied.

(j) $|x+1| + |x+6| \geq 4$

Solution:
Where are our "special" points?

$x + 1 = 0$
$x = -1$

$x + 6 = 0$
$x = -6$

Since the distance between the special points is 5, we know for all x on the number line $|x + 1| + |x + 6| \geq 5$

Since $5 \geq 4$ we know all values of x satisfy the inequality.