Problem Set 3 - Solutions

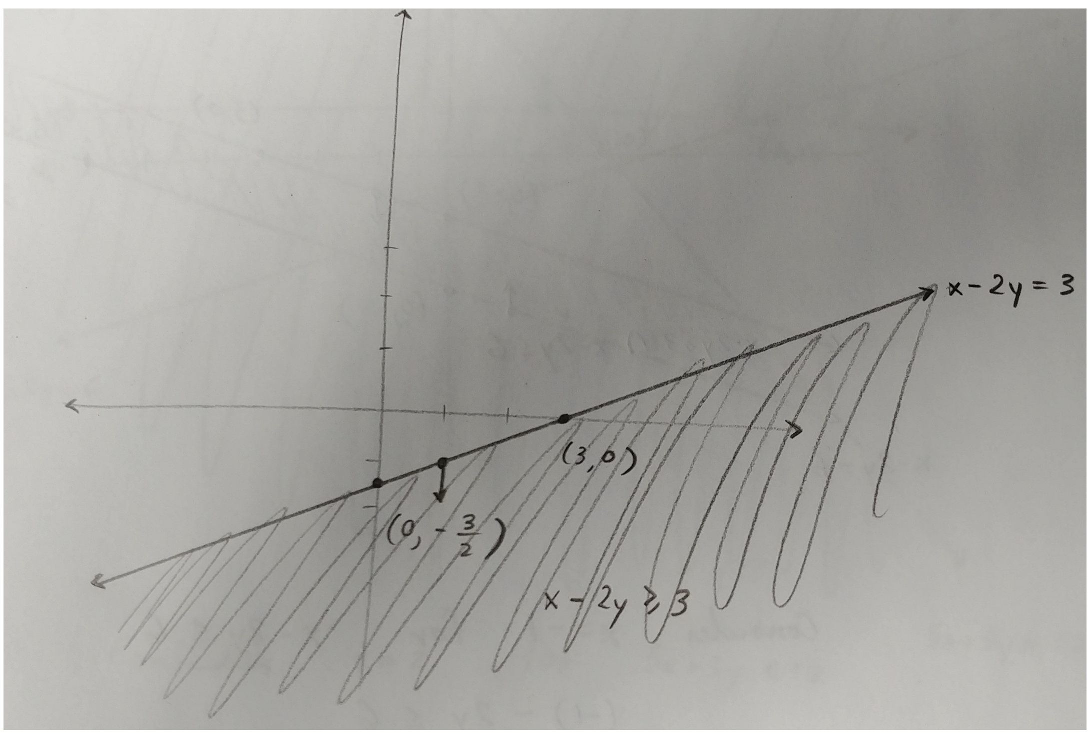
Intermediate Math Circles Fall 2019
Even More Fun With Inequalities

Nov. 13, 2019

Two Variable Linear Inequalities

Graph the following regions that satisfy the inequalities

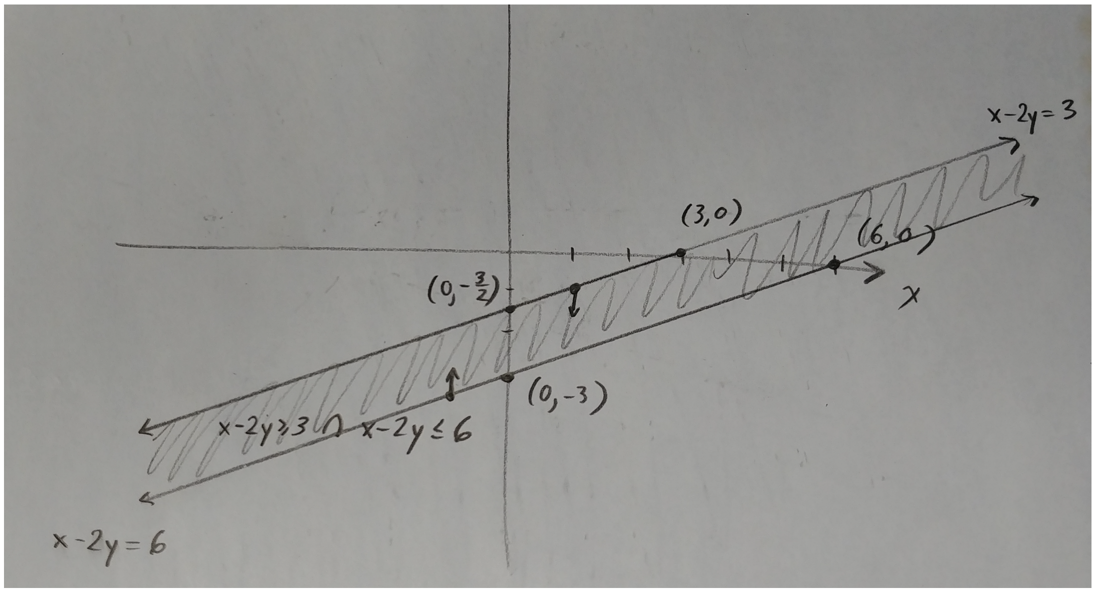
1. $x - 2y \geq 3$



Consider when x - 1 for $x - 2y \geq 3$.

$$
\begin{aligned}
1 - 2y &\geq 3 \\
1 - 3 &\geq 2y \\
-2 &\geq 2y \\
\frac{2y}{2} &\leq \frac{-2}{2} \\
y &\leq -1
\end{aligned}
$$

2. $x - 2y \geq 3 \cap x - 2y \leq 6$

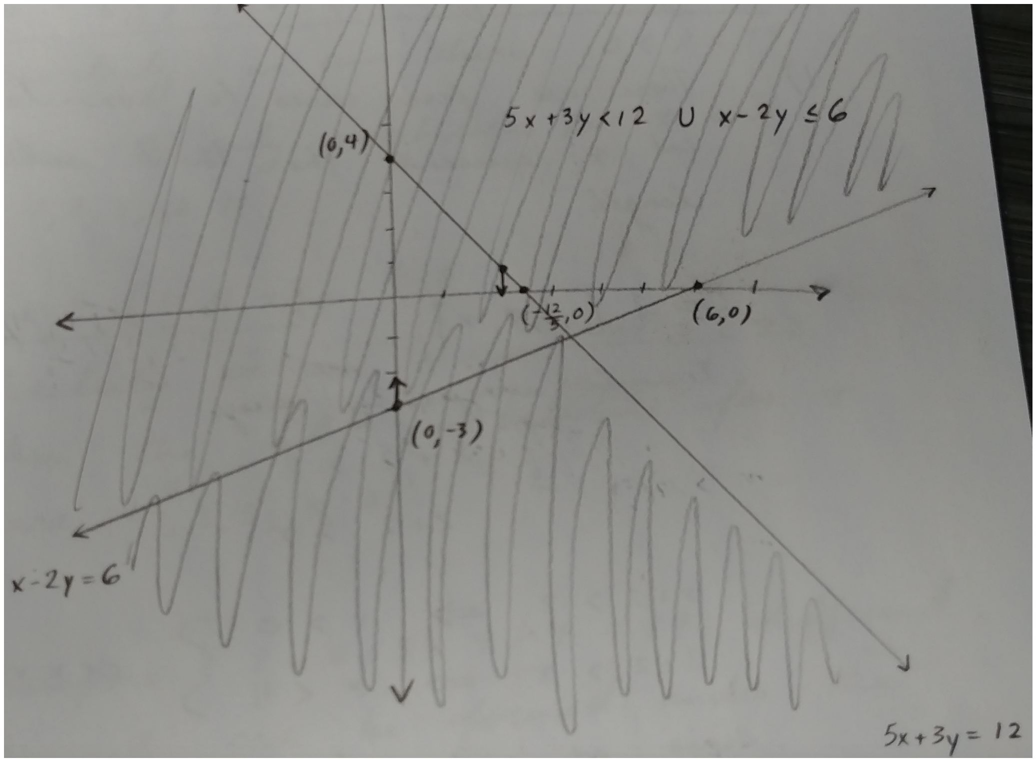


Consider x = -1 for $x - 2y \leq 6$.

$$
\begin{aligned}
(-1) - 2y &\leq 6 \\
-2y &\leq 7 \\
\frac{-2y}{-2} &\geq \frac{7}{-2} \\
y &\geq \frac{-7}{2}
\end{aligned}
$$

Note that we flip the inequality sign when dividing by -2.

3. $5x + 3y < 12 \cup x - 2y \leq 6$

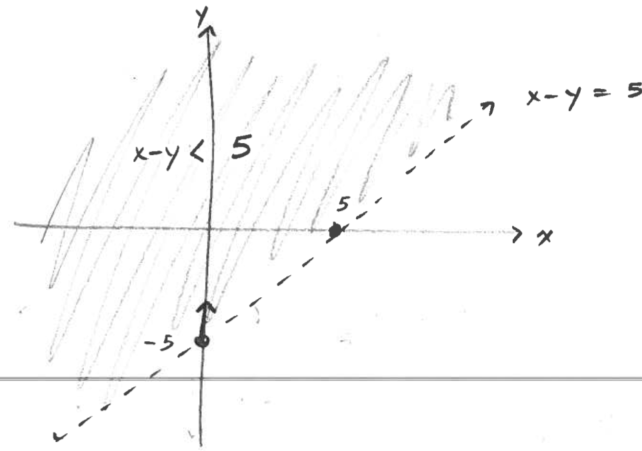


Consider x = 2 for $5x + 3y < 12$.

$$\begin{aligned} 5(2) + 3y &< 12 \\ 10 + 3y &< 12 \\ 3y &< 12 - 10 \\ y &< \frac{2}{3} \end{aligned}$$

4. $x - y < 5$

1.)



Consider when $x = 0$ for $x - y < 5$

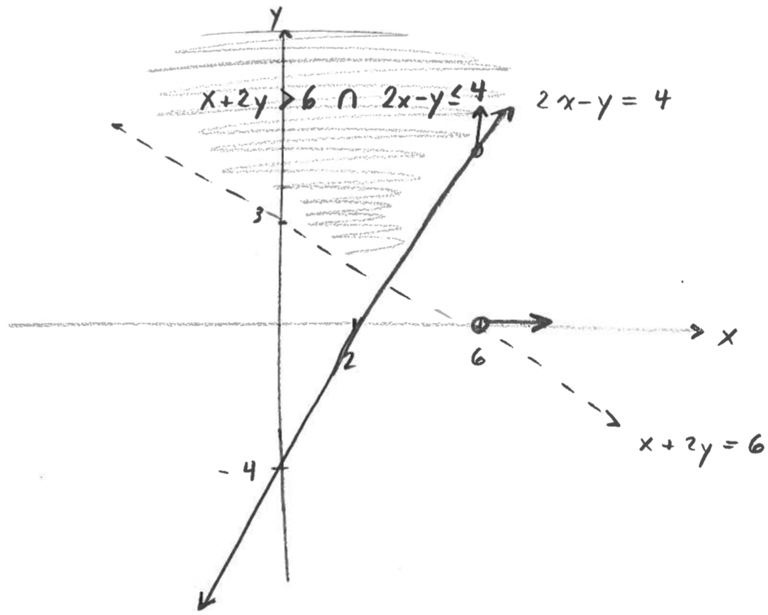
$0 - y < 5$

$-y < 5$

$y > -5$

5. $x + 2y > 6 \cap 2x - y \leq 4$

2.)



Consider $y = 0$ for $x + 2y > 6$,

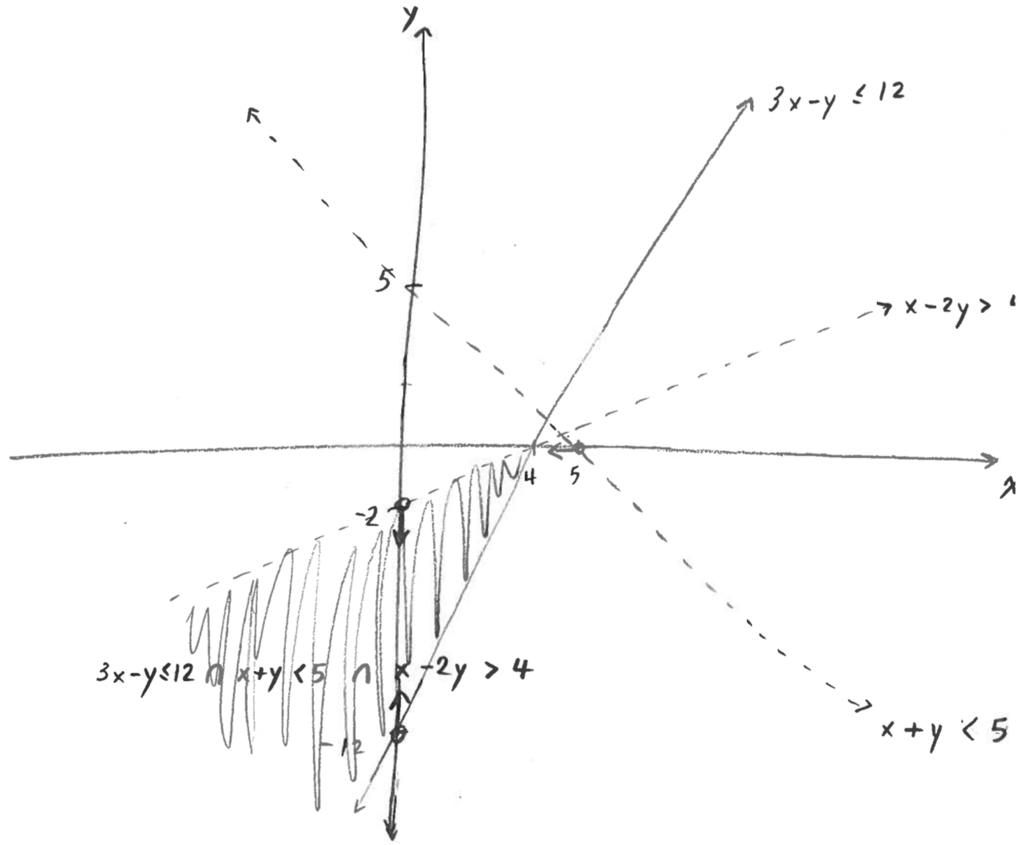
$$x + 2(0) > 6$$
$$x > 6$$

Consider $x = 6$ for $2x - y \leq 4$

$$2(6) - y \leq 4$$
$$12 - 4 \leq y$$
$$8 \leq y$$
$$y \geq 8$$

6. $3x - y \leq 12 \cap x + y < 5 \cap x - 2y > 4$

3.)



Consider $x = 0$ for $3x - y \le 12$

$3(0) - y \le 12$

$-12 \le y$

$y \ge -12$

Consider $y = 0$ for $x + y < 5$

$x + 0 < 5$

$x < 5$

Consider $x = 0$ for $x - 2y > 4$

$0 - 2y > 4$

$-4 > 2y$

$y < -2$

More Absolute Values (Review)

Solve each of the following inequalities algebraically and graphically

1. $|x-7|+|x-1|<8$

   Where are our "special" points?

   $x-7=0$ $\quad$ $x-1=0$

   $x=7$ $\quad$ $x=1$

   We know for $1<x<7$ that $|x-7|+|x-1|=6$. To make sure $|x-7|+|x-1|<8$ we can't be more than one away from 7 and one away from 1. Therefore $0<x<8$ satisfy the inequality.

Use your knowledge about absolute values to prove the following properties. *Hint: cases are your friend.*

2. If a and b are any real numbers and $b \neq 0$, then $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

   In order to prove this we need to consider 4 cases.

   Case 1 $[a \geq 0,\ b>0]$

   If $a \geq 0, b>0$ and $\frac{a}{b}>0$ we know $|\frac{a}{b}| = \frac{a}{b}$

   Since $a \geq 0$ and $b>0$ we know $|a|=a$ and $|b|=b$

   Thus $|\frac{a}{b}| = \frac{a}{b} = \frac{|a|}{|b|}$

   Case 2 $[a \geq 0, b<0]$

   If $a \geq 0, b<0$ then $\frac{a}{b} \leq 0$ and $|\frac{a}{b}| = -\frac{a}{b}$

   Since $a \geq 0$ and $b<0$ we know $|a|=a$ and $|b|=-b$

   Thus, $|\frac{a}{b}| = -\frac{a}{b} = \frac{a}{-b} = \frac{|a|}{|b|}$

   Case 3 $[a<0, b>0]$

   If $a<0$ and $b>0$, then $\frac{a}{b}<0$ and $|\frac{a}{b}| = -\frac{a}{b}$

Since $a < 0$ and $b > 0$, we know $|a| = -a$ and $|b| = b$

Thus, $|\frac{a}{b}| = -\frac{a}{b} = \frac{-a}{b} = \frac{|a|}{|b|}$

Case 4 $[a < 0, b < 0]$

If $a < 0$ and $b < 0$, then $\frac{a}{b} > 0$ and $|\frac{a}{b}| = \frac{a}{b}$

Since $a < 0$ and $b < 0$, we know $|a| = -a$ and $|b| = -b$

Thus, $|\frac{a}{b}| = \frac{a}{b} = \frac{-a}{-b} = \frac{|a|}{|b|}$

Therefore we know $|\frac{a}{b}| = \frac{|a|}{|b|}$ **when** a **and** b **are real numbers and** $b \neq 0$

3. If a is a real number and n is an integer, then $|a^n| = |a|^n$

   To prove $|a^n| = |a|^n$ we will consider the cases when $a \geq 0$ and $a < 0$
   Proof:

   Case 1 $[a \geq 0]$
   If $a \geq 0$ then $a^n \geq 0$ and $|a| = a$
   Thus $|a^n| = a^n = |a|^n$

   Case 2 $[a < 0]$
   If $a < 0$, then $|a| = -a$
   If n is even, then $a^n > 0$ and $|a^n| = a^n$
   With n even $(-1)^n = 1$
   Thus $|a^n| = a^n = (-1)^n a^n = (-a)^n = |a|^n$

   If n is odd, then $a^n < 0$ and $|a^n| = -a^n$
   With n odd $(-1)^n = -1$
   Thus $|a^n| = -a^n = (-1)a^n = (-1)^n a^n = (-a)^n = |a|^n$

   Therefore we know $|a^n| = |a|^n$ when a is a real number and n is an integer

Triangle Inequality

1. A triangle can be formed having side lengths 4, 5 and 8. It is impossible however, to construct a triangle with side lengths 4, 5 and 10. Using the side lengths 2, 3, 5, 7 and 11, how many different triangles *with exactly two equal sides can be formed*?

   There are five cases to consider. Let x represent the third side length.
   Case 1 [2,2,x]
   Triangle inequality says

   $$x + 2 > 2$$
   $$2 + 2 > x$$

   $x + 2 > 2 \Longrightarrow x > 0$
   $2 + 2 > x \Longrightarrow x < 4$
   So $0 < x < 4$.
   Since we can only have two equal side lengths x = 3 is our only possibility.
   Case 2 [3,3,x]
   Similarly we can show $0 < x < 6$.
   Thus, the only possibilities for x are 2 and 5.
   Case 3 [5,5,x]
   Similarly we can show $0 < x < 10$.
   Thus the only possibilities for x are 2, 3 and 7.
   Case 4 [7,7,x]
   Similarly we can show $0 < x < 14$.
   Thus the only possibilities for x are 2, 3, 5 and 11.
   Case 5 [11,11,x]
   Similarly we can show $0 < x < 22$.
   Thus the only possibilities for x are 2, 3, 5 and 7.
   Therefore $1 + 2 + 3 + 4 + 4 = 14$ different triangles can be formed under the given conditions.