Math Circles: Topology

Amanda Garcia

November 27, 2019

Recap

Last time, we discussed the notion of topological equivalence and explored the topological space called a torus. Today, we will continue to expand our inventory of topological spaces. Before we can add spaces to our list, we need one more concept, which we will explore in the next activity.

Exercise Use the strip provided to you and do the following:

1. Draw a simple figure which can be seen "upside down" and "rightside up" in the first cell of the strip. For example, I will choose a smiley face: ☺.
2. Draw the same figure (keeping the size and dimensions) in each remaining cell on the top side of the strip.
3. Flip your strip over, and *trace* the figure from the other side. Depending on the figure you chose, the image might be reflected.

Once your strip is complete, do the following activity:

1. Form your strip into a band. Draw the starting and ending images below.

   Starting Image

   Ending Image

   What do you notice about the images?

2. Form your strip into a band, but give it half a twist before joining the ends. Draw the starting and ending images below.

   Starting Image   Ending Image

   What do you notice about the images?

3. Form your strip into a band, but give it a full twist before joining the ends. Draw the starting and ending images below.

   Starting Image   Ending Image

   What do you notice about the images?

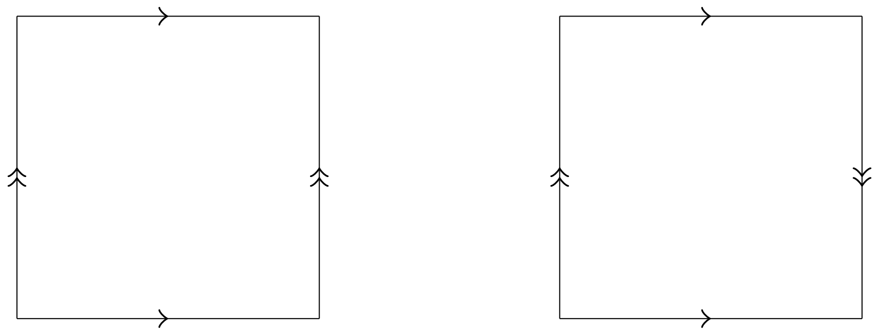
As you noticed, in some cases, the ending image would appear upside down. This brings us to an important concept in topology:

Definition. A surface is *orientable* if a figure making all possible trips along the surface does not change its orientation at any time.

Question Which of the bands formed in the above exercises are orientable surfaces?

2. Klein Bottle

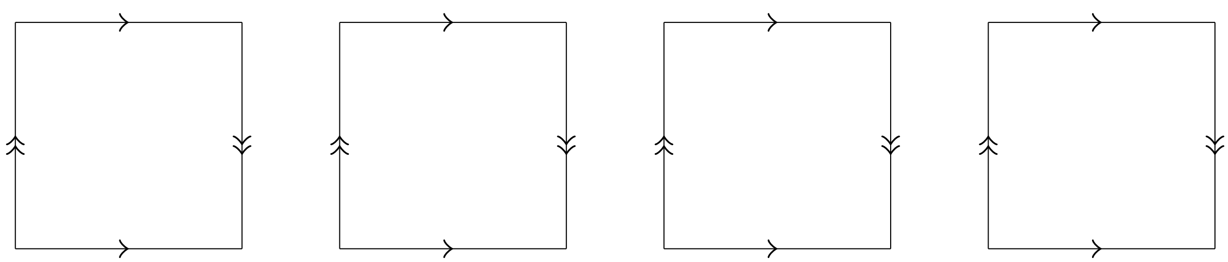
Our next topological space is called a *Klein bottle*. Let's contrast it with the torus:



Question What is the difference between the torus and the Klein bottle?

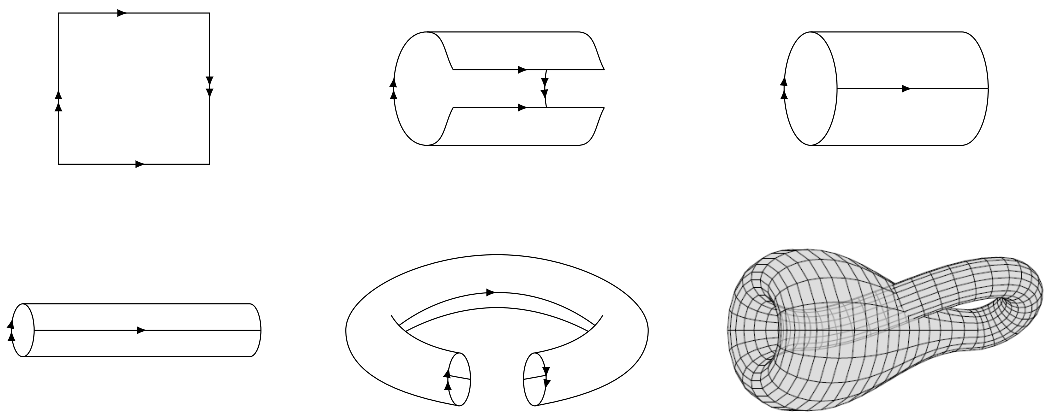
As we did with the torus, let's explore how things move along the surface of a Klein bottle.

Exercise Suppose that you have a pentagon moving along the surface of a Klein bottle. Complete the pictures below to illustrate the pentagon's movements.



Question Is the Klein bottle an orientable surface? Why or why not?

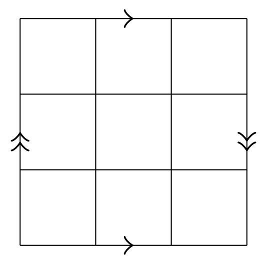
So what does a Klein bottle look like? The process of making a Klein bottle from the square is similar to that of a torus, with an additional twist!



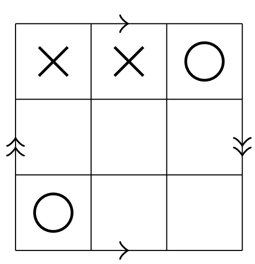
`https://www.youtube.com/watch?v=yaeyNjUPVqs`

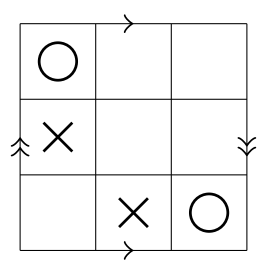
Game Let's play Klein Bottle Tic-Tac-Toe!

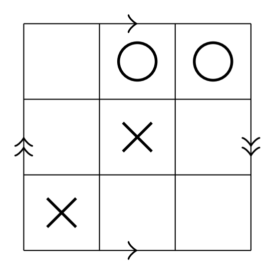
Let's start by figuring out what happens around the usual Tic-Tac-Toe board:



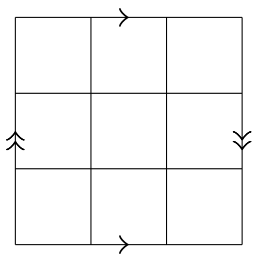
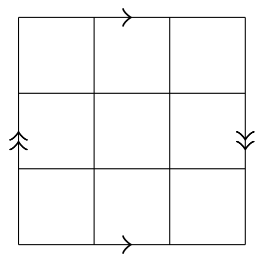
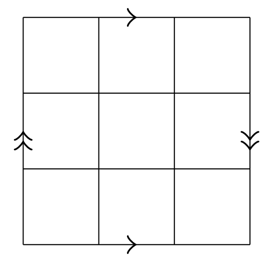
Next, let's fill in the winning move for the X player in the games below:

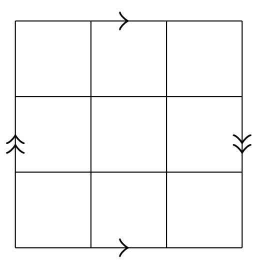
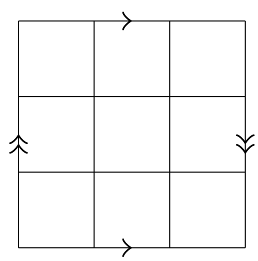
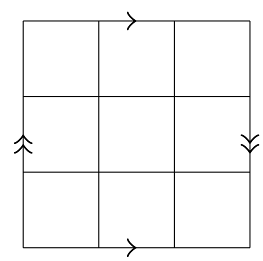


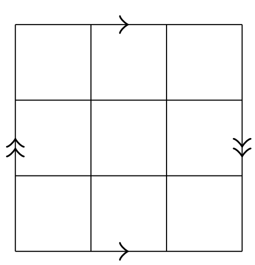
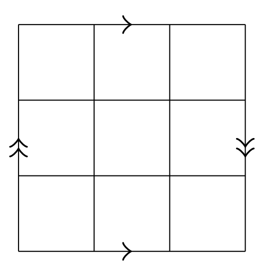
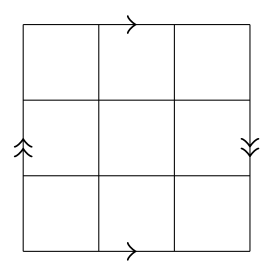




Take a few minutes to play Klein Bottle Tic-Tac-Toe with a partner.

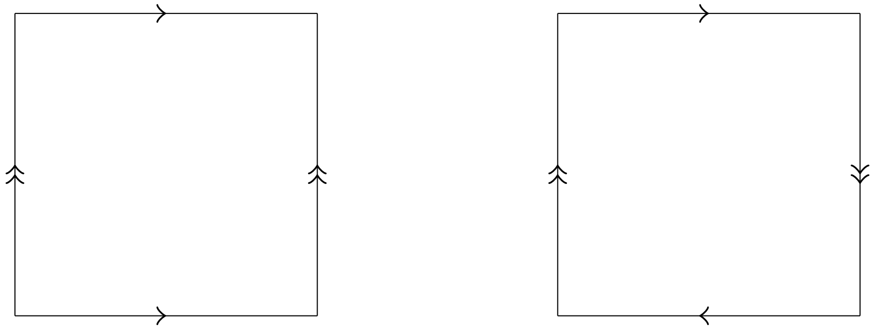




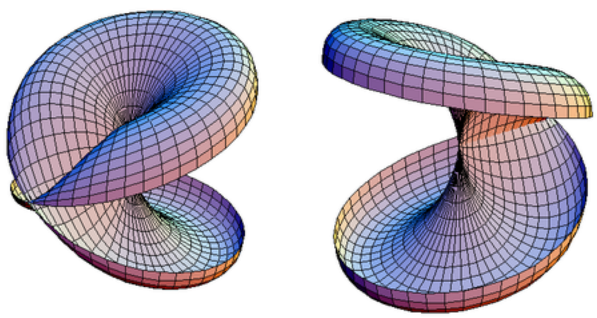




3. Projective Plane

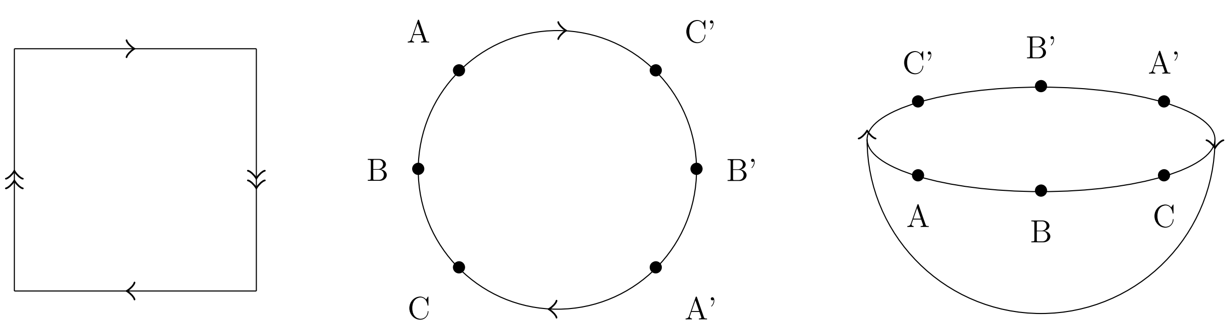
At this point, you might be wondering what kind of surface we get if we swap both pairs of arrows on the torus diagram:



The surface we end up with is called the *projective plane*. Here is a visualisation of the projective plane:



Another common way to represent the projective plane is by a hemisphere where opposite points are glued along the edge:



In the hemisphere representation, the opposite points are glued on the circular boundary together: A to A', B to B', and C to C' etc. As a figure crosses the rim, its right and left sides are switched: its orientation changes.

Surface Inventory

Let's take a moment to review some of the surfaces we have worked with:

1. Torus
2. Klein bottle
3. Projective plane
4. Sphere (from Problem Set I)

As it turns out, this list of 2-dimensional surfaces covers all of the nice™ 2-dimensional surfaces that we're interested in. In fact, we can make our list even shorter:

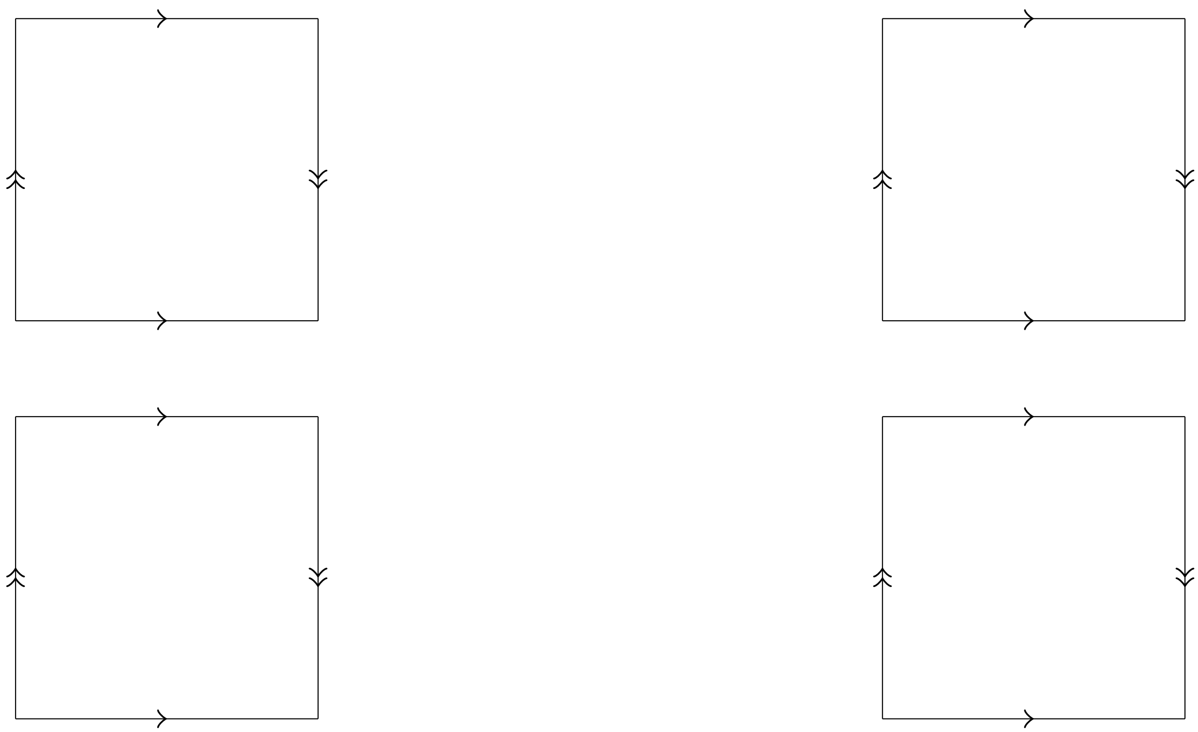
Theorem. *Any nice™ 2-dimensional surface is topologically equivalent to either:*

- *A sphere (if the surface has no holes and is orientable)*
- *A many-holed torus (if the surface has holes and is orientable)*
- *A set of projective planes connected to one another (if the surface is non-orientable)*

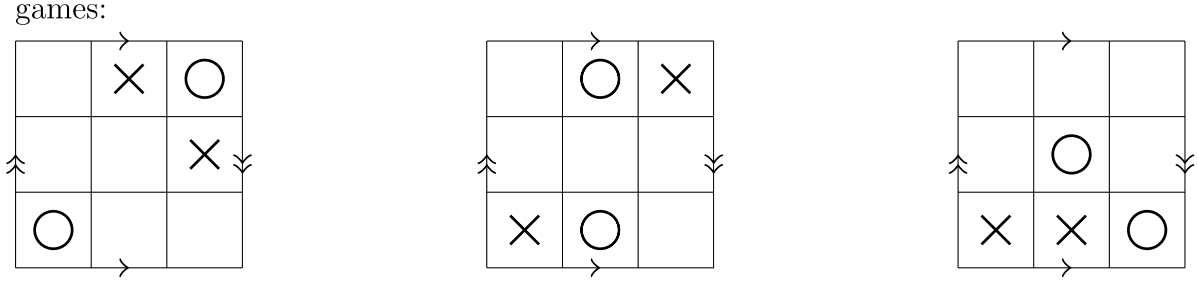
Problem Set II

1. Here's a joke about topology: "A topologist is a person who cannot tell the difference between a coffee mug and a doughnut". Explain this joke, and see if you can find a visual explanation.

2. For each Klein bottle and figure pair shown below, draw the missing part of the figure in the correct location of the Klein bottle.



3. Find the winning move for the X player in the following Klein bottle Tic-Tac-Toe games:



4. One interesting feature about Torus and Klein bottle Tic-Tac-Toe is that, unlike regular Tic-Tac-Toe, there is always a winner. Discuss why this is the case.

5. Do the following word search on a torus:

bat	pig
camel	possum
cat	puma
jaguar	rat
llama	seal
ox	shrew
panda	tapir

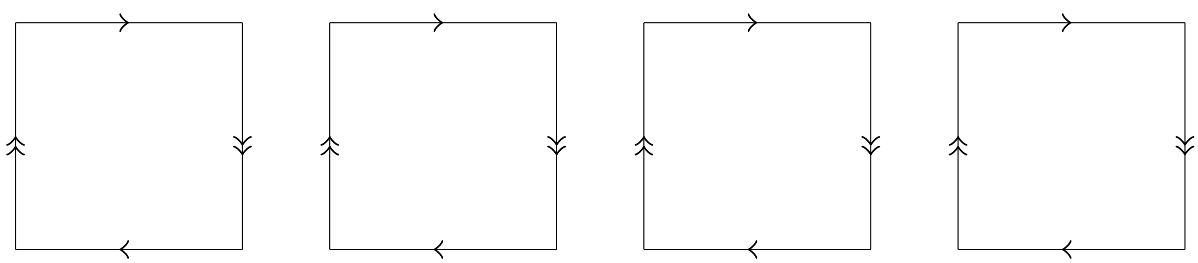
h	l	m	e	a	i
n	a	l	n	b	r
j	l	d	a	e	a
c	a	t	w	m	t
x	e	g	i	p	a
o	s	s	u	m	p

6. Do the following word search on a Klein bottle. (Hint: on a Klein bottle, letters and their mirror images are not distinguishable, so “d” and “b” are the same, and so are “p” and “q”.)

ash	lilac
birch	maple
cedar	oak
elm	palm
fir	pine
larch	poplar

h	p	m	a	c	m
a	d	e	c	k	r
a	d	n	f	e	l
a	l	i	l	m	c
s	r	q	r	e	a
h	a	h	o	c	l

7. We mentioned that the projective plane is a non-orientable surface. Try to convince yourself of this fact using its square representation and following the movement of a pentagon along the surface. You can sketch some of your observations below.



8. Suppose that a 2-dimensional polar bear lives at the south pole on a projective plane. Where on the projective plane is the bear farthest from home?

9. Suppose that two 2-dimensional polar bears live on a projective plane. The bears can move anywhere on the projective plane, but want to be as far apart as possible. How can they achieve this?

References

Word searches: `http://geometrygames.org/TorusGames/index.html`

Some problem set exercises borrowed and/or adapted from Ferron, Nathaniel, ”*An Introduction to Topology for the High School Student*” (2017). Masters Essays. 76. `http://collected.jcu.edu/mastersessays/76`