Intermediate Math Circles
October 16, 2019
Counting Part II

Upcoming Contest Dates:

CIMC Wednesday November 20

Beaver Computing Challenge Weeks of November 4 and November 11

Last week, after looking at the product and sum rules, you looked at counting permutations of objects. You first counted permutations of entire sets and ended the night by looking at the number of ways you could arrange r objects from n distinct objects where the order you selected the objects was important.

Warm-Up Problem

How many permutations of the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ taken 5 at a time:

a.) have 7 and 8 in adjacent positions

Begin by grouping 7 and 8 together as one item, giving us 7 items to arrange in a group of 5. To make sure that {7, 8} is included in the permutation, put it anywhere in the permutation (there are 4 places it can go), and then permute the 7 remaining numbers into the 3 empty spaces, which can be done in $7 \times 6 \times 5 = 210$ ways.

Using the Product Rule gives us $4 \times 210 = 840$ permutations, but remember that the grouped item {7, 8} can itself be arranged in 2 ways, so the total number of permutations is $2 \times 840 = 1680$.

b.) have 7 and 8 separated by exactly 1 number

Start by finding how many permutations with 7 appearing before 8 satisfy this condition, then multiply your answer by 2 like in part b) to include permutations with 8 appearing first.
7 is able to go in the first, second, or third position with 8 appearing in the third, fourth, or fifth position respectively. The 7 remaining digits can then be permuted in the 3 remaining spaces.

In total - using the Product Rule and brackets for clarity - there are
$3 \times (7 \times 6 \times 5) \times 2 = 1260$ possible permutations.

Today, we will look at combinations: the number of ways in which we can choose r objects from n distinct objects, where the order of selection does not matter.

Example 1:

A math student is given a list of 5 math problems and is asked to do solve any 3 of the problems. How many different problem selections can the student make?

Solution:

Let us name the problems 1, 2, 3, 4, and 5. Then, the possible choices are:

$$\begin{array}{ccccc} \{1,2,3\} & \{1,2,4\} & \{1,2,5\} & \{1,3,4\} & \{1,3,5\} \\ \{1,4,5\} & \{2,3,4\} & \{2,3,5\} & \{2,4,5\} & \{3,4,5\} \end{array}$$

Observe that since it does not matter which order the student does the problems from the list, the choice $\{1,2,3\}$ would be the same as listing the choices $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$ or $\{3,2,1\}$.

We want to determine how to count these mathematically. One way we can think about this is that we want to count all the permutations of the n objects taken r at a time, and then remove all the permutations which are the same.

In the example above, the total number of permutations is $5 \times 4 \times 3 = 60$.

For any particular permutation, how many other permutations contain exactly the same numbers? Consider the permutation $\{(1,2,3)\}$. How many other permutations contain exactly the same three numbers? This would be all the permutations on 3 objects so it is $3! = 6$.

So we have 60 total permutations, but we can organize these into groups of 6, which all contain the same objects. Since order does not matter, the total number of possible choices is 60 objects divided by 6 groups = 10. This matches what we did above when we listed all of the possibilities.

Therefore, the number of ways in which we can choose r objects from n distinct objects when order does not matter is the number of ways to arrange r objects taken from n objects when order matters divided by the number of ways to arrange r objects. So the number of combinations is

$$\frac{P(n,r)}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{(n-r)!r!}$$

In our example, $n = 5$, $r = 3$ and therefore $n - r = 2$.

So the number of ways to choose 3 questions from 5 questions is

$$\frac{P(5,3)}{3!} = \frac{5!}{2!3!} = 10.$$

Therefore, the number of ways in which we can choose r objects from n distinct objects when order does not matter is the number of ways to arrange r objects taken from n objects when order matters divided by the number of ways to arrange r objects. So the number of combinations is $\dfrac{n!}{(n-r)!r!}$

We denote this with the symbol $\dbinom{n}{r}$ and say "n choose r".

$$\therefore \binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Example 2:

At a cafeteria, a student is allowed to pick 4 different items from the following list: pop, juice, milk, water, burger, hotdog, vegetable soup, banana, orange, and apple pie.

a.) How many different choices does the student have?

Solution: The student can choose 4 items from 10 items.

$$\binom{10}{4} = \frac{10!}{6! \times 4!} = \frac{10 \times 9 \times \cancel{8} \times 7}{\cancel{4} \times 3 \times \cancel{2} \times 1} = 10 \times 3 \times 7 = 210.$$

$\therefore$The student has 210 choices for his meal.

b) How many different choices does the student have, if they don't like apple pie?

Solution: Now the student can choose 4 items from only 9 items.

$$\binom{9}{4} = \frac{9!}{5! \times 4!} = \frac{9 \times \cancel{8} \times 7 \times 6}{\cancel{4} \times 3 \times \cancel{2} \times 1} = 3 \times 7 \times 6 = 126.$$

$\therefore$The student has 126 choices for his meal.

c) How many different choices does the student have, if they must pick one and only one drink?

Solution: The student has 4 choices of drink to select from. Then, the student can pick any 3 of the remaining 6 non-drink items so has $\dbinom{6}{3}$ ways of picking those. By the product rule, the student has $4 \times \dbinom{6}{3} = 4 \times \dfrac{\cancel{6} \times 5 \times 4}{\cancel{3} \times \cancel{2} \times 1} = 4 \times 5 \times 4 = 80$ choices.

Exercises

1. Evaluate the following:

   a.) $\binom{7}{3}$ b.) $\binom{10}{2}$ c.) $\dfrac{\binom{6}{4}}{\binom{6}{2}}$

2. For a quest, the knight, Sir Cumference, needs to pick 4 out of his 10 fellow knights.

   a.) In how many ways can he do this?

   b.) In how many ways can he do this, if he must pick one particular knight, Sir Kull.

   c.) In how many ways can he do this, if he can't pick Sir Kull?

3. A student, taking a math test, is told to answer any 7 of the 10 questions. In how many ways can the student do this?

4. A high-school of 520 senior students and 480 junior students must select 5 students to represent them at a competition.

   a.) In how many ways can they do this?

   b.) In how many ways can they do this if seniors are not allowed to be selected?

   c.) In how many ways can they do this if the group must have 3 seniors and 2 juniors?

   d.) In how many ways can they do this if the group must have more seniors than juniors?

Answers:

1. a) 35 b) 45 c) 1

2. a) $\binom{10}{4} = 210$ b) $\binom{9}{3} = 84$ c) $\binom{9}{4} = 126$

3. $\binom{10}{7} = 120$

4. a) $\binom{1000}{5}$ b) $\binom{480}{5}$ c) $\binom{520}{3} \times \binom{480}{2}$

4. d) $\binom{520}{5} + \binom{520}{4} \times \binom{480}{1} + \binom{520}{3} \times \binom{480}{2}$

By examining the values of $\binom{n}{r}$ as we change n and r, a pattern emerges. Together, we will proceed to find the pattern.

We begin with $\binom{0}{0}$. This is the number of ways we can pick 0 objects from 0 objects. We are not really doing much so we will say there is just 1 way of doing this. So $\binom{0}{0} = 1$.

What is $\binom{1}{0}$? How many ways can we pick 0 objects from 1 object? There is only one way, we simply don't pick any objects. So $\binom{1}{0} = 1$. Notice then that we have $\binom{1}{0} = \dfrac{1!}{0! \times 1!} = 1$, so we define $0! = 1$.

What is $\binom{1}{1}$? How many ways can we pick 1 object from 1 object? There is only one way, we pick the object. So $\binom{1}{1} = 1$. Notice that we have $\binom{1}{1} = \dfrac{1!}{1! \times 0!} = 1$, so again we see that $0!$ must equal 1.

We calculate that $\binom{2}{0} = \dfrac{2!}{2! \times 0!} = 1$, $\binom{2}{1} = \dfrac{2!}{1! \times 1!} = 2$, $\binom{2}{2} = \dfrac{2!}{0! \times 2!} = 1$,
$\binom{3}{0} = \dfrac{3!}{3! \times 0!} = 1$, $\binom{3}{1} = \dfrac{3!}{2! \times 1!} = 3$, $\binom{3}{2} = \dfrac{3!}{1! \times 2!} = 3$, $\binom{3}{3} = \dfrac{3!}{0! \times 3!} = 1$.

Let's start making these into a table in the form of a triangle:

$$
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}
$$

Substituting in the values gives

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1
\end{array}
$$

<u>**Exercise:**</u> Fill in the next 3 rows in the above triangle. Try to find patterns in the triangle and use it to fill in 3 more rows.

Observe that the ends of each row are always 1 and the middle numbers are just the sum of the two numbers above it. This gives the triangle

$$\begin{array}{ccccccccccccc}
&&&&&&1&&&&&& \\
&&&&&1&&1&&&&& \\
&&&&1&&2&&1&&&& \\
&&&1&&3&&3&&1&&& \\
&&1&&4&&6&&4&&1&& \\
&1&&5&&10&&10&&5&&1& \\
1&&6&&15&&20&&15&&6&&1
\end{array}$$

which is called **Pascal's triangle**.

$$\begin{array}{ccccccccccccc}
&&&&&&\binom{0}{0}&&&&&& \\
&&&&&\binom{1}{0}&&\binom{1}{1}&&&&& \\
&&&&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&&&& \\
&&&\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}&&& \\
&&\binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4}&& \\
&\binom{5}{0}&&\binom{5}{1}&&\binom{5}{2}&&\binom{5}{3}&&\binom{5}{4}&&\binom{5}{5}& \\
\binom{6}{0}&&\binom{6}{1}&&\binom{6}{2}&&\binom{6}{3}&&\binom{6}{4}&&\binom{6}{5}&&\binom{6}{6}
\end{array}$$

Pascal's triangle has many interesting properties and uses. One immediate use is that it suggests some properties of $\binom{n}{r}$.

In particular, notice that $\binom{n}{r} = \binom{n}{n-r}$. But Pascal's triangle is not a proof of this fact, So let us prove it.

Proposition: For any non-negative integers n and r with $n \geq r$ we have $\binom{n}{r} = \binom{n}{n-r}$.

Proof:

We have $\binom{n}{n-r} = \dfrac{n!}{(n-(n-r))! \times (n-r)!} = \dfrac{n!}{r! \times (n-r)!} = \dfrac{n!}{(n-r)! \times r!} = \binom{n}{r}$. $\square$

The triangle also suggests another property which is much harder to prove, namely that

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}.$$

We will not prove this proposition here but will illustrate it by referring back to Exercise 1 (a): For a quest, the knight, Sir Cumference, needs to pick 4 out of his 10 fellow knights. In how many ways can he do this?

The number of ways to choose the knights was $\binom{10}{4}$.

We could have solved the problem by counting the number of groups with Sir Kull in it and adding the number of groups without Sir Kull in it.

Using the sum rule we obtain $\binom{9}{3} + \binom{9}{4}$.

$$\therefore \binom{10}{4} = \binom{9}{3} + \binom{9}{4}.$$

Exercises

1. **a.)** Mr E. Lipps has in front of him a circle, a square, and a rectangle. In how many ways can he select some of the shapes? (He may pick any number of the shapes.)

   b.) If he has 4 shapes instead of three, in how many ways can he select some of the shapes?

   c.) If he has 5 shapes instead of three, in how many ways can he select some of the shapes?

   d.) What if he had n shapes?

2. Determine $\dfrac{\binom{n}{k}}{\binom{n}{k-1}}$.

Answers:

1. a) $\binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 3 + 3 + 1 = 7 = 2^3 - 1$

1. b) $\binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4} = 4 + 6 + 4 + 1 = 15 = 2^4 - 1$

1. c) $\binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5} = 5 + 10 + 10 + 5 + 1 = 31 = 2^5 - 1$

1. d) $2^n - 1$

2. $\dfrac{n-k+1}{k}$

Combinations with Repetition

Example 3:

A pop machine offers 6 kinds of pop (Pepsi, Coke, Sprite, Ginger Ale, Dr. Pepper, and Orange). Polly Knowmeal wants to purchase 4 cans. How many different purchases can she make?

Solution 1:

- Choose all the same kind: $\binom{6}{1} = 6$ ways. PPPP
- Choose 3 of one kind and 1 of another: $\binom{6}{1} \times \binom{5}{1} = 6 \times 5 = 30$ ways. CCCS
- Choose 2 each of two kinds: $\binom{6}{2} = 15$ ways. GGOO
- Choose 2 of one kind and 1 of each of two other kinds: $\binom{6}{1} \times \binom{5}{2} = 6 \times 10 = 60$ ways. DDPO
- Choose 4 different kinds: $\binom{6}{4} = 15$ ways. PCGD

There are $6 + 30 + 15 + 60 + 15 = 126$ different purchases.

Solution 2:

Partition into 6 areas using 5 bars | | | | |. Let 4 stars * * * * represent the cans of pop. Now we can arrange 9 items, 5 bars and 4 stars in $\frac{9!}{5!4!} = \binom{9}{4} = 126$.

This method is called **Stars and Bars**.

Example 4:

Given the equation $x + y + z + w = 5$ where $x,\ y,\ z,\ w$ are non-negative integers, determine the number of different solutions.

Solution 1

- One variable has value 5, all others have value 0: $\binom{4}{1} = 4$ ways. (0,5,0,0)
- One variable has value 4, one variable has value 1, all others have value 0: $\binom{4}{1} \times \binom{3}{1} = 4 \times 3 = 12$ ways. (1,0,4,0)
- One variable has value 3, one variable has value 2, all others have value 0: $\binom{4}{1} \times \binom{3}{1} = 4 \times 3 = 12$ ways. (0,0,3,2)
- One variable has value 3, two variables have value 1, the other has value 0: $\binom{4}{1} \times \binom{3}{2} = 4 \times 3 = 12$ ways. (1,0,3,1)
- Two variables have value 2, one variable has value 1, the other has value 0: $\binom{4}{2} \times \binom{2}{1} = 6 \times 2 = 12$ ways. (0,2,1,2)
- One variable has value 2, all others have value 1: $\binom{4}{1} = 4$ ways. (2,1,1,1)

There are $4 + 12 + 12 + 12 + 12 + 4 = 56$ possible solutions.

Solution 2

Partition into 4 areas using 3 bars (| | |).
Use 5 stars (* * * * *) since we know that the sum is 5. Now we can arrange 8 items, 3 bars and 5 stars in $\frac{8!}{3!5!} = \binom{8}{5} = \frac{(8)(7)(6)(5)(4)(3)(2)(1)}{(3)(2)(1)(5)(4)(3)(2)(1)} = \frac{(8)(7)(6)}{(3)(2)(1)} = 56$ ways.

Therefore, there are 56 non negative integer solutions.