Intermediate Math Circles
October 16, 2019
Counting Part II
Problem Set Solutions

1. If 50 different students try out for a team of 30 players, in how many different ways can the coach choose the team?

   Solution

   Since the order of selection to the team does not matter, we simply choose 30 from 50 in $\binom{50}{30}$ ways.

   Some of you may have tried to determine the answer in expanded form using your calculator. The way that this answer is left is quite acceptable.

2. How many groups can be formed from 8 adults and 5 children if:

   a.) the group must have exactly 2 adults and 2 children?

   Solution

   There must be 2 adults chosen from a total of 8 adults and 2 children chosen from a total of 5 children. Using the product rule we obtain:

$$\begin{aligned}\binom{8}{2} \times \binom{5}{2} &= 28 \times 10 \\ &= 280\end{aligned}$$

   There are 280 ways to make a group that has exactly 2 adults and 2 children.

   b.) the group can be any size, but must have at least one member and an equal number of adults and children?

   Solution

   There can be a group with 1 adult and 1 child, or 2 adults and 2 children, or 3 adults and 3 children, or 4 adults and 4 children, or 5 adults and 5 children. Using the product and sum rules, we obtain:

$$\begin{aligned}&\binom{5}{1} \times \binom{8}{1} + \binom{5}{2} \times \binom{8}{2} + \binom{5}{3} \times \binom{8}{3} + \binom{5}{4} \times \binom{8}{4} + \binom{5}{5} \times \binom{8}{5} \\ =\ & (5)(8) + (10)(28) + (10)(56) + (5)(70) + (1)(56) \\ =\ & 40 + 280 + 560 + 350 + 56 \\ =\ & 1\ 286\end{aligned}$$

   There are 1 286 ways to make a group that has an equal number of adults and children.

3. How many groups containing seven different numbers can be formed by selecting the numbers from the set $\{1, 2, \ldots, 20\}$ such that

a.) 19 is the largest number in the group?

Solution

If 19 is the largest number in the group, there are 6 more numbers that need to be put into the group from the 18 numbers that remain (as 20 and 19 cannot be put into the group). So there are $\binom{18}{6}$ or 18 564 groups.

$\therefore$ there are 18 564 ways to make a group of seven from this number set so that 19 is the largest number in the group.

b.) 9 is the middle number in the group?

Solution

If 9 is the middle number in the group, this means that there are 3 numbers from the 8 numbers that are less than 9 and 3 numbers from the 11 numbers that are greater than 9. So using the product rule, there are $\binom{8}{3} \times \binom{11}{3} = (56)(165) = 9\,240$ groups.

$\therefore$ there are 9 240 ways to make a group of seven where 9 is the middle number in the group.

c.) the difference between the largest and smallest number in the group is equal to 14?

Solution

The possible pairs of largest and smallest numbers that differ by 14 are:

Largest	**Smallest**
1	15
2	16
3	17
4	18
5	19
6	20

There are 6 pairs of numbers from the set that differ by 14. For each of these pairs, there are 13 numbers in between. We must choose 5 numbers from the 13 to be put in the group. So there are $6 \times \binom{13}{5} = 6 \times 1\,287 = 7\,722$ groups.

There are 7 722 ways to make a group of seven in which the difference between the largest value and smallest value is 14.

4. With a standard deck of 52 cards, a subset of 5 cards is called a hand.

a.) How many hands are there?

Solution

Since the order of selection of the cards does not matter, we simply choose 5 cards from 52 cards in $\binom{52}{5}$ or 2 598 960 ways.

$\therefore$ there are 2 598 960 hands.

b.) How many hands contain exactly one pair? (2 of a kind and 3 different cards)

Solution

There are 13 different card values to choose for the one pair. Once the value is selected, there are four cards with the same numeric value and we can select two in $\binom{4}{2}$ ways. Once this pair is chosen, there are 12 remaining card values from which 3 different values must be chosen (giving us $\binom{12}{3}$ ways to chose our remaining 3 values). There are 4 cards (4 suits) with each value, giving us 4 ways to chose each of our 3 remaining cards. Since these four criteria must be followed at the same time, we can use the product rule.

$$\begin{aligned} 13 \times \binom{4}{2} \times \binom{12}{3} \times 4^3 &= 13 \times 6 \times 220 \times 64 \\ &= 1\ 098\ 240 \end{aligned}$$

$\therefore$ there are 1 098 240 hands that contain exactly one pair and three other cards, each with a different value.

c.) How many hands have 4 of a kind?

Solution

There are 13 different card values and we must choose one of them for our four of a kind. There is only one way to choose the four cards once the value is chosen. After choosing the four cards for our four of a kind, there are 48 cards remaining from which 1 card must be chosen for the final card.

Using the product rule, we obtain $\binom{13}{1} \times \binom{4}{4} \times \binom{48}{1} = 13 \times 1 \times 48 = 624$ different hands.

$\therefore$ there are 624 hands that contain 4 of a kind.

5. Evaluate the following **without** the use of a calculator.

a) $\dfrac{\binom{7}{3}}{\binom{7}{4}}$ b) $\dfrac{\binom{12}{8}}{\binom{9}{4}}$ c) $\dfrac{\binom{n}{3}}{\binom{n}{2}}$, $n \geq 3$.

Solution

a)

$$\frac{\binom{7}{3}}{\binom{7}{4}} = \frac{\left(\dfrac{7!}{3! \times 4!}\right)}{\left(\dfrac{7!}{4! \times 3!}\right)} = \frac{7!}{3! \times 4!} \times \frac{4! \times 3!}{7!} = 1$$

b)

$$\frac{\binom{12}{8}}{\binom{9}{4}} = \frac{\dfrac{12!}{8! \times 4!}}{\dfrac{9!}{4! \times 5!}} = \frac{12!}{8! \times 4!} \times \frac{5! \times 4!}{9!} = \frac{12 \times 11 \times 10}{8 \times 7 \times 6} = \frac{11 \times 5}{2 \times 7} = \frac{55}{14}$$

Note: $\dfrac{12!}{9!} = \dfrac{12 \times 11 \times 10 \times 9!}{9!} = 12 \times 11 \times 10$ and $\dfrac{5!}{8!} = \dfrac{5!}{8 \times 7 \times 6 \times 5!} = \dfrac{1}{8 \times 7 \times 6}$

c)

$$\frac{\binom{n}{3}}{\binom{n}{2}} = \frac{\dfrac{n!}{3! \times (n-3)!}}{\dfrac{n!}{2! \times (n-2)!}} = \frac{n!}{3! \times (n-3)!} \times \frac{2! \times (n-2)!}{n!} = \frac{n-2}{3}$$

Note: $\dfrac{(n-2)!}{(n-3)!} = \dfrac{(n-2) \times (n-3)!}{(n-3)!} = (n-2)$ and $\dfrac{2!}{3!} = \dfrac{2!}{3 \times 2!} = \dfrac{1}{3}$

6. a) How many non-negative integer solutions are there for the equation $x + y + z = 10$?

Partition into 3 areas using 2 bars $\mid \mid$.
Use 10 stars * * * * * * * * * * since we know that the sum is 10.
Now we can arrange 12 items, 2 bars and 10 stars in $\dfrac{12!}{2!10!} = \dfrac{(12)(11)}{(2)(1)} = 66$ ways.

Therefore, there are 66 non negative integer solutions.

You may wish to try to determine all possible combinations and count each. Be careful not to double count and be sure not to miss any combinations.

b) Given the equation $x + y + z + w = 20$ where x, y, z, w are integers, $x \geq -2$, $y \geq -1$, $z \geq 0$, $w \geq 1$, determine the number of different solutions.

Can we make this problem similar to the problem above?

Since $x \geq -2$, $x + 2 \geq 0$. Let $X = x + 2$. Then $X \geq 0$.
Since $y \geq -1$, $y + 1 \geq 0$. Let $Y = y + 1$. Then $Y \geq 0$.
Since $z \geq 0$ already , let $Z = z$. Then $Z \geq 0$. (Unnessesary but being consistent.)
Since $w \geq 1$, $w - 1 \geq 0$. Let $W = w - 1$. Then $W \geq 0$.

Now,

$$\begin{aligned} x + y + z + w &= 20 \\ x + 2 + y + 1 + z + w - 1 &= 20 + 2 + 1 - 1 \\ X + Y + Z + W &= 22, X, Y, Z, W \geq 0 \end{aligned}$$

Partition into 4 areas using 3 bars $\mid \mid \mid$.
Use 22 stars * since we know that the sum is 22.
Now we can arrange 25 items, 3 bars and 22 stars in $\dfrac{25!}{3!22!} = \dfrac{(25)(24)(23)}{(3)(2)(1)} = 2300$ ways.

Therefore, there are 2300 solutions.

The original problem was made simpler and then the Stars and Bars method was applied. This problem would be very difficult to solve without using Stars and Bars.