Intermediate Math Circles
October 23, 2019
Counting Part 3 - More Counting

Problem Set 1

1. How many 6-digit numbers have only odd digits?

   Solution:. There are 5 odd digits (1, 3, 5, 7, and 9). So for each digit there are 5 choices. Therefore, there are $5^6 = 15\,625$ 6-digit numbers.

2. A publisher has 3000 identical copies of a book. How many ways can they store the books at 3 different warehouses?

   Solution: We are distributing 3000 indistinguishable objects into 3 distinguishable boxes. This can be done in $\binom{3000+3-1}{3000} = \binom{3002}{3000} = 4\,504\,501$ ways.

3. How many ways are there to deal hands of 7 cards to each of five players from a standard deck of 52 cards?

   Solution: We can think of this situation as distributing 52 distinguishable objects (the 52 cards) into 6 distinguishable boxes (the 5 players plus the leftovers"box") where five of the boxes get 7 objects and one box gets 17 objects $(52 - 5 \times 7 = 17)$. This can be done in $\dfrac{52!}{(7!)^5 17!} \approx 6.97 \times 10^{34}$ ways.

4. Ankur is distributing 16 different photos to his uncle's family. He wants to give each adult twice as many photos as those given to each child. If there are 4 children and 2 adults in his uncle's family, in how many ways can he distribute the photos?

   Solution: Let x be the number of pictures distributed to each child. So each adult will get $2x$ pictures. Therefore, $2(2x) + 4x = 16$ which gives us that $x = 2$. So each child gets 2 pictures and each adult gets 4 pictures. Therefore, we are distributing 16 distinguishable objects into 6 distinguishable boxes where 4 of the boxes get 2 objects and 2 of the boxes get 4 objects. This can be done in $\dfrac{16!}{2^4(4!)^2} = 2\,270\,268\,000$ ways.

5. How many solutions are there to the equation $a + b + c + d + e = 22$

   (a) where each value is a non-negative integer?

   Solution:. Since each value is a non-negative integer we can think of this situation as distributing 22 indistinguishable objects into 5 distinguishable boxes. This can be done in $\binom{22+5-1}{22} = \binom{26}{22} = 14\,950$ ways. So there are 14 950 solutions.

   (b) where each value is an integer greater than 1?

   Solution: Since each value is an integer greater than 1, then each value must be at least 2. We can think of this as distributing 2 objects into each of the 5 boxes. So what remains to be done, is to distribute $22 - 2 \times 10 = 12$ indistinguishable objects into 5 distinguishable boxes. This can be done in $\binom{12+5-1}{12} = \binom{16}{12} = 1820$ ways. So there are 1820 solutions.

Problem Set 2

1. How many ways can you place 17 identical laptops into 5 identical bags such that each bag has at least 2 laptops?

   Solution: We begin by placing 2 laptops in each bag. So we have $17 = 2 \times 5 = 7$ laptops left to place. We have to place 7 indistinguishable objects into 5 indistinguishable bags. We will list the possibilities using the convention that the terms of the sum will be written in non-increasing order. Note, that the maximum number of terms we can have in the sum is 5 since we only have 5 bags. The following distributions are possible $7, 6+1, 5+2, 5+1+1, 4+3, 4+2+1, 4+1+1+1, 3+3+1, 3+2+2, 3+2+1+1,$ $3+1+1+1, 2+2+2+1$, and $2+2+1+1+1$. So we have a total of 13 ways.

2. How many ways can we place 7 different pictures into 5 identical folders?

   Solution:. We must distribute 7 distinguishable objects into 5 indistinguishable boxes. We use the 13 distributions from question 1 and for each we determine the number of ways to arrange the pictures. If we have k bags with the the same number of photos we need to divide by $k!$ since the bags are indistinguishable.

   - 7 - This can be done in one way.
   - $6+1$ - This can be done in $\binom{7}{1} = 7$ ways.
   - $5+2$ - This can be done in $\binom{7}{5} = 21$ ways.
   - $5+1+1$ - This can be done in $\frac{\binom{7}{5}\binom{2}{1}}{2!} = 21$ ways.
   - $4+3$ - This can be done in $\binom{7}{4} = 35$ ways.
   - $4+2+1$ - This can be done in $\binom{7}{4}\binom{3}{2} = 105$ ways.
   - $4+1+1+1$ - This can be done in $\frac{\binom{7}{4}\binom{3}{1}\binom{2}{1}}{3!} = 35$ ways.
   - $3+3+1$ - This can be done in $\frac{\binom{7}{4}\binom{4}{3}}{2!} = 70$ ways.
   - $3+2+2$ - This can be done in $\frac{\binom{7}{3}\binom{4}{2}}{2!} = 105$ ways.
   - $3+2+1+1$ - This can be done in $\frac{\binom{7}{3}\binom{4}{2}\binom{2}{1}}{2!} = 210$ ways.
   - $3+1+1+1+1$ - This can be done in $\frac{\binom{7}{3}\binom{4}{1}\binom{3}{1}\binom{2}{1}}{4!} = 35$ ways.
   - $2+2+2+1$ - This can be done in $\frac{\binom{7}{2}\binom{5}{2}\binom{3}{2}}{3!} = 105$ ways.
   - $2+2+1+1+1$ - This can be done in $\frac{\binom{7}{2}\binom{5}{2}\binom{3}{1}\binom{2}{1}}{2!3!} = 105$ ways.

   Therefore, the total number of ways is $1 + 7 + 2(21) + 3(35) + 4(105) + 70 + 210 = 855$.

3. How many random English words can you write down before you are guaranteed to have two words which have the same first letter and the same last letter?

   Solution: There are $26 \times 26 = 676$ first and last letter combinations. So you must write down 677 words to guarantee that you have two words which have the same first letter and the same last letter.

4. You are placing 27 apples into 4 baskets. Find the largest integer k such that you are guaranteed to have a basket with at least k apples in it.

   Solution: We consider $k = 7$. We want to show that we are guaranteed to have a basket with 7 apples. If all baskets had 6 or less apples, then the number of apples would be at most $6 \times 4 = 24$. But we have placed apples. Therefore, at least one of the baskets has at least 7 apples.

   Can k be larger that 7? We could possibly have a distribution of 7, 7, 7, and 6 apples. In this situation, we aren't guaranteed to have k apples in a basket if $k \geq 8$. So the largest possible value of k is 7.

5. How many ways can you place a dozen different books on four distinguishable shelves if the order of the books on the shelves matter?

   Solution:. We begin by considering the books as indistinguishable. Therefore, we have a situation where are we distributing 12 indistinguishable objects into 4 distinguishable boxes. This can be done in $\binom{12+4-1}{12}\binom{15}{12} = 455$ ways.

   For each placement of the indistinguishable books on the shelves there are 12! possible orderings of the books, if we now consider them as distinguishable.

   Therefore, in total we have $455 \times 12! = 217\,945\,728\,000$ ways to arrange the books on the shelves