Intermediate Math Circles
Fall 2019
Fun With Inequalities

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What Should Be Review

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Which of the following are true and which are false?

1. $27 < 72$
2. $-27 < 72$
3. $-27 \leq -72$
4. $27 \leq 27$

What Does $\leq$ Mean?

Less than or equal to

Given two real numbers, a and b, we know that $a \leq b$ if a is equal to b or lies to the left of b on the real number line.

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Instead we should have $-72 \leq -27$

Overview of Linear Inequalities

A statement involving the symbols '>', '<', '≥', '≤' is called an inequality. For example $5 > 3$, $x \leq 4$, $x + y \geq 9$.

(i) Inequalities which do not involve variables are called numerical inequalities. For example $3 < 8$, $5 \geq 2$.

(ii) Inequalities which involve variables are called literal inequalities. For example, $x > 3$, $y \leq 5$, $x - y \geq 0$.

(iii) An inequality may contain more than one variable and it can be linear, quadratic or cubic etc. For example, $3x - 2 < 0$ is a linear inequality in one variable, $2x + 3y \geq 4$ is a linear inequality in two variables and $x^2 + 3x + 2 < 0$ is a quadratic inequality in one variable.

(iv) Inequalities involving the symbol '>' or '<' are called strict inequalities. For example, $3x - y > 5$, $x < 3$.

(v) Inequalities involving the symbol '≥' or '≤' are called slack inequalities. For example, $3x - y \geq 5$, $x \leq 5$.

Solution of an inequality:

The value(s) of the variable(s) which makes the inequality a true statement is called its solutions. The set of all solutions of an inequality is called the **solution** set of the inequality. For example, $x - 1 \geq 0$, has infinite number of solutions as all real values greater than or equal to one make it a true statement.

9 Sets

A **set** is a collection of objects. The objects in a set are called **elements**.

Example 13. Consider the set $A = \{1, 2, 3, 4\}$.

The elements of A are 1, 2, 3 and 4. When we write

$$1 \in A$$

we just mean that 1 **is an element of** A. That is, 1 **is in** the set A.

Similarly, when we write

$$2 \in A$$

we just mean that 2 **is in** the set A.

9.1 Union and intersection of sets

Consider any sets D and E. Then

$$\begin{aligned} D \cup E &= \text{the set of elements which are in } D \textbf{ or } E \text{ (or both)} \\ &= D \textbf{ union } E, \text{ and} \end{aligned}$$

$$\begin{aligned} D \cap E &= \text{the set of elements which are in } D \textbf{ and } E \\ &= D \textbf{ intersection } E. \end{aligned}$$

Example 14. Suppose that $D = \{1, 3, 4\}$ and $E = \{2, 4, 6, 7\}$. Then

$$\begin{aligned} D \cup E &= \{1, 2, 3, 4, 6, 7\} \\ D \cap E &= \{4\} \end{aligned}$$

10 Ordering Real Numbers

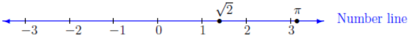
Consider any real numbers a and b.

Notation:

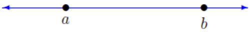
- We write $a < b$ (or $b > a$) whenever $b - a$ is positive.
- We write $a \leq b$ (or, alternatively, $b \geq a$) if $a < b$ or $a = b$.

10.1 The number line

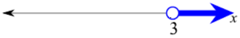
We can represent the real numbers with a number line:



If $b > a$ then b lies to the right of a on the number line:



Example 16. The set $\{x \in \mathbf{R} \mid x > 3\}$ is the set of all real numbers which lie to the right of 3 on the number line:



Example 17. The set $\{x \in \mathbf{R} \mid 1 < x \leq 3\}$ is the set of all real numbers which lie to the right of 1 and to the left of (and including) 3:

10.2 Intervals

An **interval** is a set of real numbers with "no gaps". We often denote intervals by using **round** and/or **square** brackets, as detailed below:

- A **round** bracket: (or) means that the corresponding endpoint is **not** included in the interval; that is, no "=" appears in the corresponding inequality symbol.

  On the number line this endpoint is represented by an **open** circle; that is, at the endpoint of the interval, we draw a small circle which is **not** coloured in.

In contrast,

- a **square** bracket: [or] means that the corresponding endpoint **is** included in the interval; that is, an "=" **does** appear in the corresponding inequality symbol.

  On the number line this endpoint is represented by an **closed** circle; that is, at the endpoint of the interval, we draw a small circle which **is** coloured in.

Overview

	Interval:	Bracket Notation:	Interval on the number line:
(a)	$\{x \mid a < x < b\}$	(a, b)	
(b)	$\{x \mid a \leq x \leq b\}$	$[a, b]$	
(c)	$\{x \mid a < x \leq b\}$	$(a, b]$	
(d)	$\{x \mid a \leq x < b\}$	$[a, b)$	
(e)	$\{x \mid x > a\}$	(a, ∞)	
(f)	$\{x \mid x \geq a\}$	$[a, \infty)$	
(g)	$\{x \mid x < b\}$	$(-\infty, b)$	
(h)	$\{x \mid x \leq b\}$	$(-\infty, b]$	
(i)	$\mathbb{R}$	$(-\infty, \infty)$	
(j)	$\mathbb{R}^+$	$(0, \infty)$	
(k)	$\mathbb{R}^-$	$(-\infty, 0)$	

Basic Properties

There are eight basic properties for $\leq$ and their names are in brackets on the right. For all the properties x,y,z, and r are real numbers.

(1)	$x \leq x$	(reflective)
(2)	If $x \leq y$ and $y \leq x$, then $x = y$	(antisymmetric)
(3)	If $x \leq y$ and $y \leq z$, then $x \leq z$	(transitive)
(4)	One of the following three holds: $x < y$, $y < x$, or $x = y$	(trochotomy)
(5)	If $x \leq y$, then $x + r \leq y + r$	
(6)	If $x \leq y$ and $0 \leq r$, then $rx \leq ry$	
(7)	If $x \leq y$ and $r \leq 0$, then $ry \leq rx$ $(rx \geq ry)$	
(8)	$0 \leq x^2$	

Solving Linear Inequalities (Single Variable)

Solving linear inequalities is much like solving linear equalities with one **exception**.

What's the exception?

The Exception

Remember the Property (7)

$$\text{If } x \leq y \text{ and } r \leq 0, \text{ then } ry \leq rx \ (rx \geq ry)$$

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$$\text{If } x \leq y \text{ and } r \leq 0, \text{ then } ry \leq rx \ (rx \geq ry)$$

What does this mean to multiply an inequality by a negative number?

It means that if you are reflecting your values about zero on the number line and for the inequality to still hold the values need to be flip.

The Exception

For solving inequalities, Property (7) requires us to **flip** the inequality if we multiply or divide both the left and the right hand sides by a negative number.

Example

Solve $3x < 16x - 52$

$$3x < 16x - 52$$

Is there a way to do this without multiplying or dividing by a negative?

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$$3x < 16x - 52$$
$$3x - 16x < -52$$

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$$\begin{aligned} 3x &< 16x - 52 \\ 3x - 16x &< -52 \\ -13x &< -52 \end{aligned}$$

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$$3x < 16x - 52$$
$$3x - 16x < -52$$
$$-13x < -52 \quad \leftarrow \text{ divide both sides by } -13 \text{ and flip the inequality}$$
$$\frac{-13x}{-13} > \frac{-52}{-13}$$

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$$\begin{aligned} 3x &< 16x - 52 \\ 3x - 16x &< -52 \\ -13x &< -52 \quad \leftarrow \text{ divide both sides by } -13 \text{ and flip the inequality} \\ \frac{-13x}{-13} &> \frac{-52}{-13} \\ x &> 4 \end{aligned}$$

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Solving Linear Inequalities (Single Variable)

Solve the inequality: $-\frac{1}{5}x + \frac{5}{6} \leq 5 - \frac{x}{2}$

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Solve the inequality: $-\frac{1}{5}x + \frac{5}{6} \leq 5 - \frac{x}{2}$
We clear the fractions by multiplying the LHS and the RHS by 30. We use 30 because it is the lowest common multiple.

$$\begin{aligned}
30(\frac{-x}{5} + \frac{5}{6}) &\leq 30(5 - \frac{x}{2}) \\
-6x + 25 &\leq 150 - 15x \\
-6x + 15x &\leq 150 - 25 \\
9x &\leq 125 \\
\frac{9x}{9} &\leq \frac{125}{9} \\
x &\leq \frac{125}{9}
\end{aligned}$$

Therefore $x \leq \frac{125}{9}$ satisfies the inequality.

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Using property 6 we can multiply both sides by t, where $t \geq 0$

$$t(0) \leq t(y - x)$$
$$0 \leq ty - tx$$

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$$0 + (-ty) \leq ty - tx + (-ty)$$
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Since $t \geq 0$, we know that $-t \leq 0$. Let $r = -t$.

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Since $t \geq 0$, we know that $-t \leq 0$. Let $r = -t$.
Therefore $ry \leq rx$ where $r \leq 0$. □

Word Problems (Applications)

Word Problem Solving Strategies

- Read through the entire problem
- Highlight the important information and key words that you need to solve the problem
- Identify your variables
- Write the equation or inequality
- Solve
- Write your answer in a complete sentence
- Check or justify your answer

Word Problems (Applications)

Sandy and Mandy play in the same soccer team. Last Saturday, Sandy scored 6 more goals than Mandy, but together they scored less than 16 goals. What are the possible number of goals Sandy scored?

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Assign Letters:

- the number of goals Sandy scored: S
- the number of goals Mandy scored: M

We know that Sandy scored 6 more goals than Mandy did, so: $S = M + 6$

And we know that together they scored less than 16 goals: $S + M < 16$

We are being asked for how many goals Sandy might have scored: S

SOLVE: Start with: $M + S < 16, \quad S = M + 6,$

so: $M + (M + 6) < 16$

Simplify: $2M + 6 < 16$

Subtract 4 from both sides: $2M < 16 - 6$

Simplify: $2M < 10$

Divide both sides by 2: $M < 5$

Word Problems (Applications)

Mandy scored less than 5 goals, which means that Mandy could have scored 0, 1,2, 3 or 4 goals.
Sandy scored 6 more goals than Mandy did, so Sandy could have scored 6, 7, 8, 9, or 10 goals.
Check:
- When $M = 0$, then $S = 6$ and $S + M = 6$, and $6 < 16$ is correct
- When $M = 1$, then $S = 7$ and $S + M = 8$, and $8 < 16$ is correct
- When $M = 2$, then $S = 8$ and $S + M = 10$, and $10 < 16$ is correct
- When $M = 3$, then $S = 9$ and $S + M = 12$, and $12 < 16$ is correct
- When $M = 4$, then $S = 10$ and $S + M = 14$, and $14 < 16$ is correct
- (But when $M = 5$, then $S = 11$ and $S + M = 16$, and $16 < 16$ is incorrect)

References

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- http://www.algebra-class.com/solving-word-problems-in-algebra.html
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Thank you!