Intermediate Math Circles
October 9, 2019
Counting Part 1 - Permutations

Combinatorics is the study of the arrangement of objects. It is primarily focused on the counting of these arrangements. For example, we might want to count the number of passwords that are possible using certain characters or how many different ways a student can choose their courses.

Basics of Counting - Product Rule
Suppose that a procedure can be broken down into a sequence of two tasks and that the completion of these tasks do not affect one another. If there are p ways to do the first task and for each of these ways there are q ways to do the second task, then there are $p \times q$ ways to complete the procedure.

Example 1: At Math Camp there are 4 choices of activities that are offered in the morning. There are 3 choices of activities that are offered in the afternoon. How many ways can Simon organize his day at camp?

$$4 \times 3 = 12 \ \ ways$$

Example 2:. How many 2 letter acronyms are possible using the letters of the English alphabet?
There are 26 letters in the English alphabet so 26 choices for the first letter and 26 choices for the second letter.

$$26 \times 26 = 676 \ \ acronyms$$

Product Rule Extended
The product rule can be extended to a procedure with more than two tasks. We simply calculate the product of the number of ways to complete each of the tasks that make up the procedure.

Example 1: How many 7-character passwords can be formed using only numbers and English letters?
There are $26 \times 2 = 52$ possible letter choices (uppercase and lowercase) and 10 digits from 0 to 9 so there are 62 possible choices for each character of the password.

$$62^7 \approx 3.5216 \times 10^{12} \ \ possible \ \ passwords$$

Example 2: A new company has 4 new employees and 7 empty offices. How many ways are there to assign a different office to each of these 4 employees?
The first employee can be given one of the 7 offices, the second employee can be given one of the remaining 6, etc.

$$7 \times 6 \times 5 \times 4 = 840 \ \ ways$$

Basics of Counting - Sum Rule
If a task can be done in either one of p ways or in one of q ways (where none of the set of p ways is the same as any of the q ways), then there are $p + q$ ways to do the task.

Example 1: A student has to choose a course. There are 7 math courses and 5 physics courses to choose from. How many ways can the student choose the course?

$$7 + 5 = 12 \ \ courses$$

Example 2: How many 2-digit numbers end in 3 or 7?
There are 9 ways for the digit to end in 3 (since we can't have 0 as the first digit) and there are 9 ways for the digit to end in 7.

$$9 + 9 = 18 \ \ total \ \ ways$$

Sum Rule Extended
The sum rule can be extended to a task that can be done in more than two sets of ways. We simply sum the number of ways in each of these sets.

Example 1: How many 3-digit numbers end in 1, 2, or 3?
There are $9 \times 10 \times 1 = 90$ 3-digit numbers that end in 1 since 0 can't be the first digit and the last digit must be 1 and there are 10 choices for the second digit. Similarly, there are 90 3-digit numbers that end in 2 and 90 3-digit numbers that end in 30.

$$90 + 90 + 90 = 270 \ \ total \ \ ways$$

Example 2: How many 3-digit numbers begin with a 1 or end with a 4 or a 6?
There are $1 \times 10 \times 10 = 100$ ways to begin with a 1 and from the previous example, 90 ways to end with a 4 and 90 ways to end with a 6 so there are $100 + 90 + 90 = 280$ ways. But we have **double counted** some numbers which start with 1 and end with a 4 or 6. There are $1 \times 10 \times 1 = 10$ ways to start with 1 and end with 4 and similarly, there are 10 ways to start with 1 and end in 6.

$$280 - 20 = 260 \ \ numbers$$

Principle of Inclusion-Exclusion
If a task can be done in p ways or q ways, the number of ways to do the task is $p + q$ minus the number of ways to do the task that are common to the different ways.
Example 1: How many positive integers less than 100 are divisible by either 3 or 7?
There are 33 positive integers divisible by 3, 14 divisible by 7 and 4 divisible by 3 **and** 7 (divisible by 21).

$$33 + 14 - 4 = 43 \ \ positive \ \ integers$$

Example 2: At a school with 1240 students, 879 are taking a math course, 385 are taking a music course and 167 are taking neither a math course nor a music course. How many students are taking both a math course and a music course?
Let x be the number of students taking both a math course and a music course.

$$\begin{aligned} 879 + 385 - x &= 1240 - 167 \\ 1264 - x &= 1073 \\ x &= 191 \ \ students \end{aligned}$$

Problem Set 1

1. How many positive integers less than 1000 are
   (a) divisibly by 3 or 4?
   (b) divisible by 3 but not by 4?
2. How many 7-character license plates are possible using only letters and numbers? How many are possible if no character is repeated?
3. A palindrome is the same when read forwards and backwards.
   (a) How many 4-digit numbers are palindromes?
   (b) How many 4-digit numbers are palindromes and divisible by 4?
4. How many strings of 6 English letters contain exactly one vowel? (Consider y as a consonant.)
5. In how many ways can a photographer at a wedding arrange 6 people in a row, including the bride and groom, if
   (a) the bride must be next to the groom?
   (b) the bride is not next to the groom?
   (c) the bride is positioned somewhere to the left of the groom?

Permutations
Example 1: How many ways can we order 5 different books on a shelf?
There are 5 available books for the first position, 4 remaining books for the second position, etc.

$$5 \times 4 \times 3 \times 2 \times 1 = 120 \ \ ways$$

Example 2: If we have 10 different books to choose from, how many ways can we order 5 books on a shelf?
There are 10 available books for the first position, 9 remaining books for the second position, etc.

$$10 \times 9 \times 8 \times 7 \times 6 = 30,240 \ \ ways$$

A **permutation** of a set of distinct objects is an ordered arrangement of the objects.

An ordered arrangement of r elements from a set of distinct objects is called an r-permutation.
Factorial
For an integer $n \geq 0$, n factorial, denoted $n!$, is defined by

$$0! = 1$$

$$n! = n(n-1)(n-2)\cdots(3)(2)(1) \text{ for } n \geq 1.$$

Examples:

1. 3! $\quad 3 \times 2 \times 1 = 6$
2. 8! $\quad 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$
3. 13! $\quad 13 \times 12 \times 11 \times ... \times 3 \times 2 \times 1 = 6,227,020,800$

Counting Permutations
The number of permutations of n objects is $n!$.

The number of r-permutations from a set of n distinct objects is
$n(n-1)(n-2)...(n-r+1) = \dfrac{n!}{(n-r)!}$.

Example 1: How many ways can 7 people stand in a straight line?
7 choices of people can be first in line. One of the remaining 6 can be second. etc.

$$7! = 5,040$$

Example 2: If 200 people enter a contest, how many ways can first, second and third prize be awarded?
The first prize is chosen first and can be given to one of the 200 people. The second prize can be given to one of the remaining 199. etc.

$$200 \times 199 \times 198 = 7,880,400$$

More Examples

1. How many permutations of the letters PANDEMIC contain

   (a) the string AD?

   Consider **AD** as one object. So we arrange the remaining 6 letters (P,D,E,M,I,C) and the one object **AD** so we are arranging 7 objects.

   $$7! = 5,040$$

   (b) the strings ME and NAP?

   Similar to part (a), we arrange **ME**, **NAP**, D, I and C, so 5 objects.

   $$5! = 120$$

2. How many 6-digit numbers have no repeated digits?

   The first digit cannot be a 0 so there are 9 choices. The second digit can be a 0 but we've already used one digit and cannot repeat it so there are 9 choices for the second digit. There are 8 remaining choices for the third digit, etc.

   $$9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136,080$$

3. How many permutations of the letters of AGRICOLA are there?

   There are 8 letters but A appears twice so we must avoid double counting.

   $$\frac{8!}{2!} = \frac{40,320}{2} = 20,160$$

4. How many permutations of the letters BANANAGRAMS are there?

   There are 11 letters but A appears 4 times and N appears twice so we must avoid double counting.

   $$\frac{11!}{4! \times 2!} = 831,600$$

Permutations With Repetition

The number of permutations of n objects where there are n_1 indistinguishable type 1 objects, n_2 indistinguishable type 2 objects, $\ldots$, n_k indistinguishable type k objects, where $n_1 + n_2 + \cdots + n_k = n$ is $\dfrac{n!}{n_1! \times n_2! \times \ldots \times n_k!}$

Example: How many ways can you order the letters of MISSISSIPPI?
There are 11 letters in total but S appears 4 times, I appears 4 times and P appears twice so we must avoid double counting.

$$\frac{11!}{4! \times 4! \times 2!} = 34,650$$

Problem Set 2

1. How many ways can you order the letters of TARAMASALATA?
2. One hundred tickets are sold to 100 different people for a draw. Four different prizes are awarded, including a grand prize. How many ways are there to award these prizes if
   (a) there are no restrictions?
   (b) Bob wins the grand prize?
   (c) Bob wins one of the prizes?
   (d) Bob does not win a prize?
   (e) Bob and Jen both win prizes?
   (f) Bob and Jen both win prizes, but Al and Ed do not?
3. How many 5-digit numbers have exactly two 3s?
4. How many ways are there for 8 men and 5 women to stand in a line so that no two women stand next to each other?