Intermediate Math Circles
October 9, 2019
Counting Part 1 - Permutations

Problem Set 1

1. How many positive integers less than 1000 are

   (a) divisibly by 3 or 4?

   Solution: There are 333 positive integers less than 1000 which are divisible by 3. There are 249 positive integers less than 1000 which are divisible by 4. There are 83 positive integers less than 1000 which are divisible by 3 and 4 (in other words they are divisible by 12). Therefore, there are $333 + 249 - 83 = 499$ integers less than 1000 which are divisible by 3 or 4.

   (b) divisible by 3 but not by 4?

   Solution: There are 333 positive integers less than 1000 which are divisible by 3 and there are 83 positive integers less than 1000 which are divisible by 3 and 4 (in other words they are divisible by 12). Therefore, there are $333 - 83 = 250$ integers less than 1000 which are divisible by 3, but not divisible by 4.

2. How many 7-character license plates are possible using only letters and numbers? How many are possible if no character is repeated?

   Solution:. There are 36 characters available (10 digits and 26 letters). Therefore, for each of the 7 characters there are 36 choices. Therefore, there are $7^{36} \approx 7.8364 \times 10^{10}$ possible license plates if we allow characters to be repeated. If you are not allowed to repeat characters then there will be $36 \times 35 \times 34 \times 33 \times 32 \times 31 \times 30 \approx 4.2072 \times 10^{10}$ possible license plates.

3. A palindrome is the same when read forwards and backwards.
   Solution:

   (a) How many 4-digit numbers are palindromes?

   Solution:. If we choose the last two digits, then this will determine the first two digits because the number is a palindrome. The last digit of the number cannot be a 0 because this would result in the first digit being a 0 which is not possible. Therefore, we have 9 choices for the last digit and 10 choices for the second-last digit. Therefore, there are $9 \times 10 = 90$ 4-digit numbers which are palindromes.

   (b) How many 4-digit numbers are palindromes and divisible by 4?

   Solution: A 4-digit number is divisible by 4 if its last two digits when considered as a 2-digit number is divisible by 4. As we said in part (a), once we have chosen the last two digits this will determine the number because it is a palindrome. There are 24 positive integers less than 100 which are divisible by 4. Since the first digit of the 4-digit number cannot be 0, then the last digit of the number cannot be 0. So we can't choose any of these 24 integers which end in 0. The 2-digit numbers which end in 0 and are divisible by 4 are 20, 40, 60, and 80. Therefore, there are $24 - 4 = 20$ 4-digit numbers which are palindromes and divisible by 4.

4. How many strings of 6 English letters contain exactly one vowel? (Consider y as a consonant.)

   Solution:. The vowel can appear in any of the 6 positions in the string. There are 5 choices for the vowel (a, e, i, o, u). For each of the remaining 5 letters in the string we can choose one of the 21 consonants. Therefore, there are $6 \times 5 \times 21^5 = 122\,523\,030$ strings.

5. In how many ways can a photographer at a wedding arrange 6 people in a row, including the bride and groom, if

   (a) the bride must be next to the groom?

   Solution:. We consider the bride and groom as one object, so in total we have 5 objects to arrange. There are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways to arrange these objects. But for each one of these arrangements we can have the bride to the right of the groom or the bride to the left of the groom. Therefore, we have $120 \times 2 = 240$ arrangements.

   (b) the bride is not next to the groom?

   Solution:. The total number of ways to arrange 6 people is $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ ways. So we simply subtract the number of ways where the bride and groom are next to one another to get $720 - 240 = 480$ ways where the bride and groom are not next to one another.

   (c) the bride is positioned somewhere to the left of the groom?

   Solution: A very simple solution recognizes the fact that in half of all possible arrangements the bride is to the left of the groom and in the other half of the arrangements the bride is to the right of the groom. Therefore, the bride is to the left of the groom in $720 \div 2 = 360$ arrangements.

   An alternate solution is to consider the placement of the bride and groom. Once they are placed, then there are $4 \times 3 \times 2 \times 1 = 24$ ways to arrange the remaining 4 people. We number the positions from left to right. If the bride is in the first position, then there are 5 choices for the position of the groom. If the bride is in the second position, then there are 4 choices for the position of the groom. If the bride is in the third position, then there are 3 choices for the position of the groom. If the bride is in the fourth position, then there are 2 choices for the position of the groom. If the bride is in the fifth position, then there is 1 choice for the position of the groom. If the bride is in the sixth position, then there are 0 choices for the position of the groom. Therefore, there are $5 + 4 + 3 + 2 + 1 + 0 = 15$ ways that we can arrange the bride and groom. Therefore, in total there are $15 \times 24 = 360$ arrangements where the bride is to the left of the groom.

Problem Set 2

1. How many ways can you order the letters of TARAMASALATA?

   Solution:. There are two T's, 6 A's and 1 of each of the remaining letters. There are 12 letters in total. Therefore, we can order the letters in $\dfrac{12!}{2!6!} = 332\,640$ ways.

2. One hundred tickets are sold to 100 different people for a draw. Four different prizes are awarded, including a grand prize. How many ways are there to award these prizes if

   (a) there are no restrictions?

   Solution:. There are 100 choices for the grand prize, 99 choices for the first prize, 98 choices for the second prize and 97 choices for the third prize. Therefore, there are $100 \times 99 \times 98 \times 97 = 94\,109\,400$ ways to award the prizes.

   (b) Bob wins the grand prize?

   Solution: There is 1 choice for the grand prize (because it goes to Bob), 99 choices for the first prize, 98 choices for the second prize and 97 choices for the third prize. Therefore, there are $99 \times 98 \times 97 = 941\,094$ ways to award the prizes.

   (c) Bob wins one of the prizes?

   Solution:. There are 4 choices for Bob's prize. There are $99 \times 98 \times 97$ ways to award the remaining 3 prizes. Therefore, there are $4 \times 99 \times 98 \times 97 = 3\,764\,376$ ways to award the prizes.

   (d) Bob does not win a prize?

   Solution: Without Bob there are 99 choices for the grand prize, 98 choices for the first prize, 97 choices for the second prize and 96 choices for the third prize. Therefore, there are $99 \times 98 \times 97 \times 96 = 90\,345\,024$ ways to award the prizes.

   (e) Bob and Jen both win prizes?

   Solution: There are 4 choices for Bob's prize and then there are 3 choices for Jen's prize. Therefore, there are a total of 4×3 ways to decide which prizes go to Bob and Jen. There will be 98×97 ways we can award the remaining two prizes. Therefore, there are $4 \times 3 \times 98 \times 97 = 114\,072$ ways to award the prizes.

   (f) Bob and Jen both win prizes, but Al and Ed do not?

   Solution: There are 4 choices for Bob's prize and then there are 3 choices for Jen's prize. Therefore, there are a total of 4×3 ways to decide which prizes go to Bob and Jen. There will be 96×95 ways we can award the remaining two prizes since Al and Ed did not win prizes. Therefore, there are $4 \times 3 \times 96 \times 95 = 109\,440$ ways to award the prizes.

3. How many 5-digit numbers have exactly two 3s?

   Solution: We cannot have 0 in the first digit so we will consider two separate cases. The case where the first digit is a 3 and the case where the first digit is not a 3.

   In the case where the first digit is a 3, there are 4 choices for the position of the other 3. For each of the remaining three digits there are 9 choices. Therefore, we have $4 \times 9^3 = 2916$ numbers in the first case.

   In the case where the first digit is not a 3, we have 4 choices for the position of the first 3 and 3 choices for the position of the second 3. Since these 3s are identical we get

$\dfrac{4 \times 3}{2} = 6$ ways we can position the 3s. There are 8 choices for the first digit of the number (because it cannot be a 3 or a 0) and there are 9 choices for each of the remaining two digits. Therefore, we have $6 \times 8 \times 9^2 = 3888$ numbers in the second case.

Therefore, in total we have $2916 + 3888 = 6804$ 5-digit numbers with exactly two 3s.

4. How many ways are there for 8 men and 5 women to stand in a line so that no two women stand next to each other?

   Solution:. We begin by arranging the men relative to each other. We can do this in 8! ways. We now assign a position to each of the 5 women. There are 7 possible positions between two men, as well as the position to the left of the leftmost man, and the position to the right of the rightmost man. So there are 9 possible positions for the women but we an only assign at most one woman to each position. Therefore, we can assign positions to the women in $9 \times 8 \times 7 \times 6 \times 5 = 15\,120$ ways. Therefore, in total we can arrange the people in $8! \times 15\,120 = 609\,638\,400$ ways.