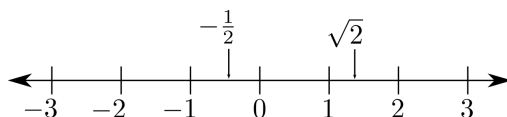# Grade 6 Math Circles

November 12/13 2019

A Number of Numbers

Different Types of Numbers

Natural Numbers ℕ

Although they used hieroglyphics to do it, the Egyptians were the first to count, so they are credited with the first numerical system.

The most basic set of numbers that we use is called the **natural numbers.** This is the set of the counting numbers {1,2,3,4,5,...}, or the set of positive whole numbers.

For example, 10000, 7, and 54 are all natural numbers, but -3, $\frac{1}{3}$ and 0.6 are not.

Whole Numbers

Zero took much longer to become common. The number is used to represent "nothingness". Mathematicians struggled with the concept of "nothing" being "something" for hundreds of years. As a result, Zero was not used as a number until around the year 500.

The set of whole numbers is the set of positive counting numbers including 0: {0,1,2,3}

Integers $\mathbb{Z}$

Long before 0 was considered a number, negative numbers were used. The concept of negativity began in China in 200 BC when doing taxes, when they wanted to cancel out the amount sold with the amount spent to show a deficit. In India, negative numbers were used to show debts.

The set of numbers {...,-3,-2,-1,0,1,2,3,...} is known as the **integers**.
For example, -72, 42 000 and 5 are integers, but -5.7 and $\frac{8}{5}$ are not.

Think: Would we say that the number **7** is a Natural Number, a Whole Number, or an Integer?

Rational Numbers $\mathbb{Q}$

So far, we have looked at whole numbers and their opposites as well as 0 . However, we have not looked at the numbers in between the whole numbers. It was the Indians in 458 who first used the decimal system, when working in astronomy. Of course, you know that a decimal can be converted into a fraction.

Now that we add in fractions, we have a whole new set. We call the **rational numbers** the set of numbers that can be written as fractions. This means that we can write a rational number as a a fraction with integers as the numerator and denominator (0 can never be the denominator of a fraction!).
Examples of rational numbers include $\frac{1}{3}$, $\frac{7}{6}$, $\frac{-2}{5}$ and $\frac{9}{1} = 9$.

Think: Would **7** be considered a Rational Number?

Irrational Numbers $\overline{\mathbb{Q}}$

The Greek mathematician Pythagoras believed that all numbers were rational. However, one day, one of Pythagoras' students, Hippasus, proved him wrong when he found that $\sqrt{2}$ could not be written as a fraction.

When a number is not rational, it cannot be written as a fraction. In terms of decimals, this means that it is never-ending! Notice that there is no pattern to the decimals of $\sqrt{2}$. If your calculator was bigger, it would show a sequence of many more digits that would never end. We call the numbers that are not rational the **irrational numbers**, denoted by $\overline{\mathbb{Q}}$.

Examples of irrational numbers include π, $\sqrt{3}$, and the Golden Ratio.

However, numbers like $\dfrac{1}{3}$, which is equal to 0.333333333 (never-ending), are rational because the never-ending decimal is a repeating number. Only never-ending random decimals are irrational.

All square roots of Whole Numbers numbers are irrational, except if the number is a perfect square.

Real Numbers $\mathbb{R}$

The set of real numbers, denoted by $\mathbb{R}$, is defined to be any number on the number line, or the set of rational and irrational numbers.

Examples of real numbers include 11, 0, -42, 0.5, $\dfrac{9}{10}$ and $\sqrt{8}$.

Think: You are told a number is Natural. What other types can you say it is for **certain**?

Think: You are told a number is Irrational. What other types can you say it is for **certain**?

Practice

Determine which sets the following numbers belong to:

a) 4

b) 4.5

c) -4

d) $\dfrac{-1}{4}$

e) π

f) $\sqrt{4}$

Factors and Multiples

1. What is a factor?

2. What is a multiple?

Locker Problem

One hundred students are assigned lockers 1 to 100. The student assigned to locker 1 opens every locker. The student assigned to locker 2 then closes every other locker. The student assigned to locker 3 changes the status of all lockers whose numbers are multiples of 3 (If a locker that is a multiple of 3 is open, the student closes it. If it is closed, the student opens it). The student assigned to locker 4 changes the status of all lockers whose numbers are multiples of 4, and so on for all 100 lockers.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1. Which students touched locker 20?

2. How can you determine if a locker is left open?

3. Which lockers will be left open? Why are they left open?

4. Which lockers were touched exactly twice?

Prime Numbers

A **prime number** is a natural number greater than 1 that only has 1 and itself as factors. It cannot be formed by multiplying a pair of smaller natural numbers.

For example: {2,3,5,7,11}

Exercise:

Find all pairs of prime numbers between 1 and 40 that add to an odd number.
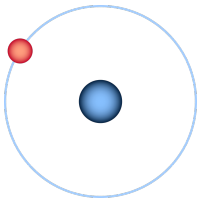
Scale of Numbers

Some of the numbers we come across when doing calculations in Math or Science are extremely large or extremely small. This makes numbers very bothersome to write out.

For example:

The mass of the sun is approximately 1,989,000,000,000,000,000,000,000,000,000 kilograms.

The mass of an electron is approximately 0.00000000000000000000000000000911 kilograms.

Writing these numbers would be very time consuming and make calculations very hard to follow. We use **Scientific Notation** to express numbers that are too big or too small to be conveniently written in "normal form."

Review: Exponents

Applying an exponent to a number means to multiply the number by itself the amount of times equal to the exponent. For example:

$$\text{exponent / power} \downarrow$$

$$2^3 = 2 \times 2 \times 2 = 8$$

$$\text{base} \nearrow$$

The exponent 3 is applied to the base 2. This means to evaluate 2^3 we calculate the **base number** multiplied by itself **exponent** times.In this case 2 multiplied by itself 3 times or $2 \times 2 \times 2 = 8$

Exercise: Powers (Exponents) of 10

Calculate the following powers of 10. Look for a pattern in your answers.

a) 10^0 b) 10^1 c) 10^2

d) 10^3 e) 10^6 f) 10^{10}

g) What is the pattern we can notice working with powers of 10?

Working with Negative Exponents/Powers:

Negative exponents work very similar to positive exponents. Consider the fact that all rational numbers can be expressed as a fraction.For example, 2 can also be written as $\frac{2}{1}$.

A negative exponent performs the same operation as with a positive exponent, just on the opposite side of the fraction line. For example:

$$\frac{2^{-3}}{1} = \frac{1}{2^3} = \frac{1}{2 \times 2 \times 2} = \frac{1}{8}$$

Exercise: Negative Powers of 10

Calculate the following powers of 10. Look for a pattern in your answers. The pattern created by powers of 10 makes them very useful in easily expressing large numbers.

a) 10^{-1}

b) 10^{-2}

c) 10^{-3}

d) 10^{-5}

e) 10^{-10}

f) What is the pattern we can notice working with negative powers of 10?

Multiplying by Powers of 10

Using the patterns earlier we see something interesting when multiplying numbers by powers of 10. For example:

$$5 \times 10^2 = 5 \times 100 = 500 \qquad\qquad 5 \times 10^{-2} = \frac{5}{100} = 0.05$$

There is a distinct pattern we can notice:

When multiplying by **positive** powers of 10 the decimal place moves to the **right** the same amount of spaces as the power.

When multiplying by **negative** powers of 10 the decimal place moves to the **left** the same amount of spaces as the power.

With this knowledge we can express any number in terms of powers of 10. We can call this **scientific notation**:

Scientific Notation has the form: N $\times 10^n$

N is a number between 1 and 10.

n is a positive or negative integer.

Now we can better understand some of the extreme numbers of our universe.

For example: 1.72×10^4

To find the number in "normal form" we look at the power of 10 and move the decimal place accordingly. For 10^4 we move the decimal place in **1.72** four places to the right.

$$1.72 \times 10^4 = 17,200$$

Exercises: Express the following numbers in "normal form"

a) 1×10^6 b) 3×10^{-2} c) 3.21×10^5

d) 1.26×10^3 e) 7.55×10^{-3} f) 1.16×10^{10}

Extreme Numbers

1. **Distances in the Universe**

    The size of the world outside our world might suprise you...

    - The earth is located 1.5×10^{11} metres away from the sun

    - The earth is located 3.84×10^5 metres away from the moon

        To put that in to perspective the sun is about 400,000 times further then the moon.

    - A lightyear is how far light travels in the timespan of 1 year: 9.461×10^{15} metres

2. **Massive Planets and Stars:**

    - Our big blue earth weighs in at $5.972 \times 10^{24} kg$

    - The sun has a mass of $1.989 \times 10^{30} kg$ The sun is almost 334,000 times as heavy as the earth!

3. **Not So Massive:**

    - **The Mass of an Electron**

        – The mass of an electron 'm_e' is about 9.11×10^{-31} kg.

        – Electrons are thought to be '*fundamental particles*', meaning that they cannot be broken down into smaller pieces.

- **The Mass of a Proton**

  - The mass of a proton 'm_p' is about $1.67 \times 10^{-27} \, kg$.

  - Protons, along with neutrons are the tiny particles which make up the nuclei of atoms!

  - A single proton has about as much mass as 1,830 electrons!

4. **A Googolplex**

   - The classic example of a massive number is a googol, which is equal to 10^{100} or a 1 followed by 100 zeros. The classic example of a ridiculously large number is a googolplex, which is a 1 followed by a googol of zeros ($10^{a \, googol}$), or $10^{10^{100}}$.

Can we even imagine how big of a number this is? Lets try to!

A googolplex has 10^{100} zeros. If we say each of these zeros can be written in a space of 1 centimeter, how long would the entire number have to be? It would have to be _____ long!

Example 4:

How long would the digits of a googolplex be in terms of the following units:

(a) Meters?

(b) Kilometers?

(c) Lightyears (The distance light can travel in an entire year)?

Problem Set:

1. Your friend tells you he has found a real natural number that is also an integer. Is this possible?

2. What type of number is the sum of two integers **guaranteed** to be? What about the sum of two Natural Numbers?

3. You are told a number is Rational. What other types can you say it is for **certain**?

4. A **factorial** denoted with an exclamation point "!", is an operation applied to a natural number. The operation calculates the product of the natural number and all natural numbers before that down to 1. You can also find 0! but let's ignore that for now.) For example:
   $4! = 4 \times 3 \times 2 \times 1 = 24$

   (a) Find 8!
   (b) Will the result of a factorial always be the same type(s)?

5. Why is 2 the only even prime number?

6. Glenda, Helga, Ioana, Julia, Karl, and Liu participated in the 2017 Canadian Team Mathematics Contest. On their team uniforms, each had a different number chosen from the list 11, 12, 13, 14, 15, 16. Helga's and Julia's numbers were even. Karl's and Liu's numbers were prime numbers. Glenda's number was a perfect square. What was Ioana's number? (Source: 2018 Pascal (Grade 9), #12)

7. The sum of two different prime numbers is 10. What is the product of these two numbers? (Source: 2007 Pascal (Grade 9), #13)

8. Express the following numbers in "normal form" :

a) 1×10^6

b) 3×10^{-2}

c) 10×10^5

d) Speed of light:

3×10^8 m/s

e) Distance to mars:

2.25×10^9 metres

f) Radius of oxygen atom:

$6 \times 10^{-11} metres$

Scientific Notation On a Scientific Calculator:

Scientific Notation has the form: $N \times 10^n$

N is a number between 1 and 10.

n is a positive or negative integer.

To enter a number in scientific notation into your calculator you can do either of the following sets of steps:

1. Enter the number N.
2. Press the multiplication button. **OR**
3. Enter the number 10.
4. Hit the exponent button ˆ .
5. Enter the power n.

1. Enter the number N.
2. Hit the "EXP" button.
3. Enter the power n.

Try doing the following questions 9 and 10 with the help of the calculator:

9. The speed of an object is the ratio of its distance and time. Thus to find the time an object takes to travel we can multiply the speed it is travelling by the distance it travels:

$$time = distance \div speed$$

   (a) The distance to the Sun is 1.5×10^{11} m and the speed of light is 3×10^8 m/s, how long does it take the light from the sun to reach Earth in minutes? (Your answer will be in seconds.)

   (b) The distance around the earth is about 4.1×10^7 m and light travels at a speed of 3×10^8 m/s. How long does it take light to tavel around the Earth? (Your answer will be in seconds.)

10. As part of his famous theory of relativity Albert Einstein stated that anything with mass has an equivalent amount of energy. This is better stated with the famous equation $E = mc^2$ where E is the equivalent energy of the object, m is the mass of the object, and c is the speed of light $3 \times 10^8 m/s$. Using this famous equation to find the equivalent energy of a box of mass 10kg.

11. The number $\sqrt{2}$ is irrational. Is the number $\dfrac{1}{\sqrt{2}}$ also irrational?

12. P, Q, R, S, T and are five different integers between 2 and 19 inclusive.

- P is a two-digit prime number whose digits add up to a prime number.

- Q is a multiple of 5.

- R is an odd number, but not a prime number.

- S is the square of a prime number.

- T is a prime number that is also the mean (average) of P and Q.

  Which number is the largest? (Source: 2007 Pascal (Grade 9), #21)

13. Challenge Question: Two different 2-digit positive integers are called a reversal pair if the position of the digits in the first integer is switched in the second integer. For example, 52 and 25 are a reversal pair. The integer 2015 has the property that it is equal to the product of three different prime numbers, two of which are a reversal pair. Including 2015, how many positive integers less than 10,000 have this same property? (Source: 2015 Gauss (Grade 7), #25)