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Grade 6 Math Circles

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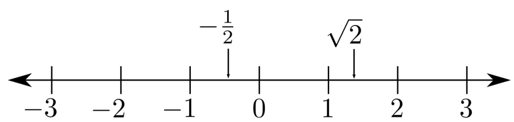
A Number of Numbers Solutions



Different Types of Numbers

Think: Would we say that the number **7** is a Natural Number, a Whole Number, or an Integer?

7 is a Natural Number, a Whole Number, and an Integer.

Think: Would **7** be considered a Rational Number?

7 could be considered as a rational number. 7 can be written as $\frac{7}{1}$ or $\frac{14}{2}$ for example. These are two fractions that also represent 7. Thus we can say 7 is a rational number.

Think: You are told a number is Natural. What other types can you say it is for **certain**?

Whole Number, Integer, Rational, Real

Think: You are told a number is Irrational. What other types can you say it is for **certain**?

Real

Practice

Determine which sets the following numbers belong to:

a) 4 Natural, Integer, Whole Number, Rational, Real

b) 4.5 Rational, Real

c) -4 Integer,Rational,Real

d) $\frac{-1}{4}$ Rational, Real

e) π Irrational, Real

f) $\sqrt{4} = 2$
Rational, Natural, Integer, Whole, Real

Factors and Multiples

1. What is a factor?

   Numbers we can multiply to get another number. 4 and 5 are factors of 20. $4 \times 5 = 20$

2. What is a multiple?

   Number that can be divided by another number without a remainder. 20 is a multiple of 5 and 4.

Locker Problem

One hundred students are assigned lockers 1 to 100. The student assigned to locker 1 opens every locker. The student assigned to locker 2 then closes every other locker. The student assigned to locker 3 changes the status of all lockers whose numbers are multiples of 3 (If a locker that is a multiple of 3 is open, the student closes it. If it is closed, the student opens it). The student assigned to locker 4 changes the status of all lockers whose numbers are multiples of 4, and so on for all 100 lockers.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

1. Which students touched locker 20?

   All the factors of 20: 1, 2,4,5,10,20

2. How can you determine if a locker is left open?

   A locker with an even number of factors will be left closed. A locker with an odd number of factors will be left open. For example 5 has two factors, 1 and 5. The first student will close the locker and the fifth student will close it. 9 has three factors: 1,3, and 9 so the first student opens the door, the third student closes the door, and the ninth student opens it again.

3. Which lockers will be left open?

   1,4,9,16,25,36,49,64,81,100 are the lockers left open. These numbers have an odd number of factors. It is no coincidence they are perfect squares! These numbers all have an odd number of factors because they all have a "pair of factors" which is a single number. For example 16 has a "pair of factors" 4 and 4 ($4 \times 4 = 16$) but we only count four as a single factor.

4. Which lockers were touched exactly twice?

   Lockers touched exactly twice are the numbers with only two factors. These two factors are 1 and the locker number itself. For example locker 5 is only touched by student 1 and student 5. We call these "prime numbers" as they are only divisible by themself and one.

   The list of lockers touched twice are:

   2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97

Prime Numbers

Exercise:

Find all pairs of prime numbers between 1 and 40 that add to an odd number.

The trick to this problem is to realize the pattern. If I add two odd numbers I will always get an even number. For example $3 + 3 = 6$. The only way that the sum of two numbers will be odd is if one of the numbers is odd and another is even. Since 2 is the only even prime number, all the pairs of prime numbers must contain 2. The set of answers will just be 2 paired with all the other prime numbers less than 40.

The set of answers is: (2,3) (2,5) (2,7) (2,11) (2,13) (2,17) (2,19) (2,23) (2,29) (2,31) (2,37)

Scale of Numbers

Exercise: Powers (Exponents) of 10

Calculate the following powers of 10. Look for a pattern in your answers.

a) $10^0 = 1$ b) $10^1 = 10$ c) $10^2 = 100$

d) $10^3 = 1000$ e) $10^6 = 1,000,000$ f) 10^{10} =10,000,000,000

g) What is the pattern we can notice working with powers of 10?

The number of zeros after the 1 is equal to the power applied.

Working with Negative Exponents/Powers:

Exercise: Negative Powers of 10

Calculate the following powers of 10. Look for a pattern in your answers. The pattern created by powers of 10 makes them very useful in easily expressing large numbers.

a) $10^{-1} = \frac{1}{10} = 0.1$ b) $10^{-2} = \frac{1}{100} = 0.01$ c) $10^{-3} = \frac{1}{1000} = 0.001$

d) $10^{-5} = \frac{1}{100,000} = 0.00001$ e) $10^{-10} = \frac{1}{10,000,000,000} = 0.0000000001$

f) What is the pattern we can notice working with negative powers of 10?

The power on the 10 is how many zeros appear before the 1.

Exercises: Express the following numbers in "normal form"

a) $1 \times 10^6 = 1{,}000{,}000$

b) $3 \times 10^{-2} = 0.03$

c) $3.21 \times 10^5 = 321{,}000$

d) $1.26 \times 10^3 = 1{,}260$

e) $7.55 \times 10^{-3} = 0.00755$

f) $1.16 \times 10^{10} = 11{,}600{,}000{,}000$

Extreme Numbers

1. **A Googolplex**

   - The classic example of a massive number is a googol, which is equal to 10^{100} or a 1 followed by 100 zeros. The classic example of a ridiculously large number is a googolplex, which is a 1 followed by a googol of zeros ($10^{a\ googol}$), or $10^{10^{100}}$.

   Can we even imagine how big of a number this is? Lets try to!

   A googolplex has 10^{100} zeros. If we say each of these zeros can be written in a space of 1 centimeter, how long would the entire number have to be? It would have to be <u>a googol centimeters</u> long!

Example 4:

How long would the digits of a googolplex be in terms of the following units:

(a) Meters?

$$10^{100}\ centimeters \times \frac{1\ meter}{100\ centimeters} = 10^{98}\ meters$$

(b) Kilometers?

$$10^{98}\, meters \times \frac{1\, kilometer}{1,000\, meters} = 10^{95}\, kilometers$$

(c) Lightyears (The distance light can travel in an entire year)?

$1\ Lightyear = 9.4608 \times 10^{15}\ meters$

Now we know a googolplex is 10^{98} meters from part (a), so now just need to see how many lightyears it takes to make 10^{98} meters.

$$10^{98}\, meters \times \frac{1\, lightyear}{9.4608 \times 10^{15}\, meters} = 1.057 \times 10^{82}\, lightyears$$

This would be the same as saying 10 billion, billion, billion, billion, billion, billion, billion, billion, billion lightyears! In comparison, the entire observable universe is a mere 93 billion lightyears.

Problem Set:

1. Your friend tells you he has found a real natural number that is also an integer. Is this possible?

   Yes this is possible. 4 for example is an integer and natural.

2. What type of number is the sum of two integers **guaranteed** to be? What about the sum of two Natural Numbers?

   If I sum two integers I am garunteed to get back an integer which will also be Rational and Real as a result. I am not garunteed to get back a natural or whole number because I could add two negative integers together and get back a negative number which would not be Whole or Natural. For example $(-4) + (-4) = -8$.

   Adding two natural numbers however I am garunteed to get back a Natural Number (which is also Whole, an Integer, Rational, and Real) because The only possible result from adding positive "whole numbers" is another positive "whole number". For example $3 + 3 = 6$ $Natural + Natural = Natural$.

3. You are told a number is Rational. What other types can you say it is for **certain**?

   A Rational Number **could** be an Integer like 7, but it could also be a fraction such as $\frac{3}{4}$. Thus the only think we can say for certain is that the number is also Real as well as Rational.

4. A **factorial** denoted with an exclamation point "!", is an operation applied to a natural number. The operation calculates the product of the natural number and all natural numbers before that down to 1. (You can also find 0! but let's ignore that for now.) For example: $4! = 4 \times 3 \times 2 \times 1 = 24$

   (a) Find $8! = 40320$

   (b) Will the result of a factorial always be the same type(s)?

   A factorial will always be calculating the product of natural numbers. Through examples we can see that the product of two natural numbers is another natural number. For example: $6 \times 6 = 36$. Thus we can say the result of a factorial will always be a natural number which is also a Whole number, an Interger, Rational, and Real.

5. Why is 2 the only even prime number?

The number two is prime as the only two numbers it could possibly have as factors are 1 and itself. Thus it is by definition a prime number. All other even numbers are divisible by 2 and thus have atleast one factor other than themselves and one. Therefore even numbers other than two cannot be prime.

6. Glenda, Helga, Ioana, Julia, Karl, and Liu participated in the 2017 Canadian Team Mathematics Contest. On their team uniforms, each had a different number chosen from the list 11, 12, 13, 14, 15, 16. Helga's and Julia's numbers were even. Karl's and Liu's numbers were prime numbers. Glenda's number was a perfect square. What was Ioana's number? (Source: 2018 Pascal (Grade 9), #12)

Of the given uniform numbers,

- 11 and 13 are prime numbers
- 16 is a perfect square
- 12, 14, and 16 are even

Since Karl's and Liu's numbers were prime numbers, then their numbers were 11 and 13 in some order. Since Glenda's number was a perfect square, then her number was 16. Since Helga's and Julia's numbers were even, then their numbers were 12 and 14 in some order. (The number 16 is already taken.) Thus, Ioana's number is the remaining number, which is 15.

7. The sum of two different prime numbers is 10. What is the product of these two numbers? (Source: 2007 Pascal (Grade 9), #13)

The prime numbers smaller than 10 are 2, 3, 5, and 7. The two of these numbers which are different and add to 10 are 3 and 7. The product of 3 and 7 is $3 \times 7 = 21$.

8. Express the following numbers in "normal form" :

a) 1×10^6 = 1,000,000

b) $3 \times 10^{-2} = 0.03$

c) $10 \times 10^5 = 1,000,000$

d) Speed of light:
3×10^8 m/s
= 300,000,000 m/s

e) Distance to mars:
2.25×10^9 metres
=2,250,000,000 metres

f) Radius of oxygen atom:
$6 \times 10^{-11} metres$
=0.00000000006 metres

Try doing the following questions 9 and 10 with the help of the calculator:

9. The speed of an object is the ratio of its distance and time. Thus to find the time an object takes to travel we can multiply the speed it is travelling by the distance it travels:

$$time = distance \div speed$$

(a) The distance to the Sun is 1.5×10^{11} m and the speed of light is 3×10^8 m/s, how long does it take the light from the Sun to reach Earth in minutes? (Your answer will be in seconds.)
speed $= 3 \times 10^8$ m/s and distance $= 1.5 \times 10^{11}$ m
$time = distance \div speed = 1.5 \times 10^{11}$ m $\div$ 3×10^8 m/s $= 500$ seconds
500 seconds ≈ 8.3333 minutes
Thus it takes light from The sun ≈ 8.33 minutes to reach Earth.

(b) The distance around the Earth is about 4.1×10^7 m and light travels at a speed of 3×10^8 m/s. How long does it take light to tavel around the Earth? (Your answer will be in seconds.)

speed $= 3 \times 10^8$ m/s and distance $= 4.1 \times 10^7$ m

$time = distance \div speed = 4.1 \times 10^7 \text{ m} \div 3 \times 10^8 \text{ m/s} \approx 0.1367$ seconds Light travels around the earth in just 0.1367 seconds. It travels around Earth more than 7 times in one second!

10. As part of his famous theory of relativity Albert Einstein stated that anything with mass has an equivalent amount of energy. This is better stated with the famous equation $E = mc^2$ where E is the equivalent energy of the object, m is the mass of the object, and c is the speed of light $3 \times 10^8 m/s$. Using this famous equation to find the equivalent energy of a box of mass 10kg.

    Mass=10kg and c $=3 \times 10^8 m/s$

    $Energy = mass \times c^2 = 10\text{kg} \times (3 \times 10^8 \text{m/s})^2 = 9 \times 10^{17}$

    Your answer is actually in the unit Joules which is used to measure energy! 9×10^{17} Joules of equivalent energy in a 10kg box.

11. The number $\sqrt{2}$ is irrational. Is the number $\frac{1}{\sqrt{2}}$ also irrational?

    Without the mathematical proof, if we are not able to express $\sqrt{2}$ (which is also $\frac{\sqrt{2}}{1}$) as a fraction of integers there is no reason to believe that we could express $\frac{1}{\sqrt{2}}$ as a fraction of integers. $\sqrt{2}$ is a number of forever repeating decimals with no pattern and flipping it to $\frac{1}{\sqrt{2}}$ would be the same. Thus we can argue that $\frac{1}{\sqrt{2}}$ is also irrational.

12. P, Q, R, S, T and are five different integers between 2 and 19 inclusive.

    - P is a two-digit prime number whose digits add up to a prime number.
    - Q is a multiple of 5.
    - R is an odd number, but not a prime number.
    - S is the square of a prime number.
    - T is a prime number that is also the mean (average) of P and Q.

      Which number is the largest? (Source: 2007 Pascal (Grade 9), #21)

Let us first consider the possibilities for each integer separately:

- The two-digit prime numbers are 11, 13, 17, 19. The only one whose digits add up to a prime number is 11. Therefore, P = 11.
- Since Q is a multiple of 5 between 2 and 19, then the possible values of Q are 5, 10, 15.
- The odd numbers between 2 and 19 that are not prime are 9 and 15, so the possible values of R are 9 and 15.
- The squares between 2 and 19 are 4, 9 and 16. Only 4 and 9 are squares of prime numbers, so the possible values of S are 4 and 9.
- Since P = 11, the possible values of Q are 5, 10 and 15, and T is the average of P and Q, then T could be 8, 10.5 or 13. Since T is also a prime number, then T must be 13, so Q = 15.

We now know that P = 11, Q = 15 and T = 13. Since the five numbers are all different, then R cannot be 15, so R = 9. Since R = 9, S cannot be 9, so S = 4. Therefore, the largest of the five integers is Q = 15.

13. Challenge Question: Two different 2-digit positive integers are called a reversal pair if the position of the digits in the first integer is switched in the second integer. For example, 52 and 25 are a reversal pair. The integer 2015 has the property that it is equal to the product of three different prime numbers, two of which are a reversal pair. Including 2015, how many positive integers less than 10,000 have this same property? (Source: 2015 Gauss (Grade 7), #25)

    All 2-digit prime numbers are odd numbers, so to create a reversal pair, both digits of each prime must be odd (so that both the original number and its reversal are odd numbers). We also note that the digit 5 cannot appear in either prime number of the reversal pair since any 2-digit number ending in 5 is not prime. Combining these two facts together leaves only the following list of prime numbers from which to search for reversal pairs: 11, 13, 17, 19, 31, 37, 71, 73, 79, and 97. This allows us to determine that the only reversal pairs are: 13 and 31, 17 and 71, 37 and 73, and 79 and 97. (Note that the reversal of 11 does not produce a different prime number and the reversal of 19 is 91, which is not prime since $7 \times 13 = 91$.)

Given a reversal pair, we must determine the prime numbers (different than each prime of the reversal pair) whose product with the reversal pair is a positive integer less than 10 000. The product of the reversal pair 79 and 97 is $79 \times 97 = 7663$. Since the smallest prime number is 2 and $2 \times 7663 = 15\ 326$, which is greater than 10 000, then the reversal pair 79 and 97 gives no possibilities that satisfy the given conditions. We continue in this way, analyzing the other 3 reversal pairs, and summarize our results in the table below.

Prime Number	Product of the Prime Number with the Reversal Pair			
	13 and 31	17 and 71	37 and 73	79 and 97
2	$2 \times 13 \times 31 = 806$	$2 \times 17 \times 71 = 2414$	$2 \times 37 \times 73 = 5402$	greater than 10 000
3	$3 \times 13 \times 31 = 1209$	$3 \times 17 \times 71 = 3621$	$3 \times 37 \times 73 = 8103$	
5	$5 \times 13 \times 31 = 2015$	$5 \times 17 \times 71 = 6035$	greater than 10 000	
7	$7 \times 13 \times 31 = 2821$	$7 \times 17 \times 71 = 8449$		
11	$11 \times 13 \times 31 = 4433$	greater than 10 000		
13	can't use 13 twice			
17	$17 \times 13 \times 31 = 6851$			
19	$19 \times 13 \times 31 = 7657$			
23	$23 \times 13 \times 31 = 9269$			
29	greater than 10 000			
Total	8	4	2	0

In any column, once we obtain a product that is greater than 10 000, we may stop evaluating subsequent products since they use a larger prime number and thus will exceed the previous product. In total, there are $8 + 4 + 2 = 14$ positive integers less than 10 000 which have the required property.