UNIVERSITY OF
WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

November 19/20

Circle Geometry

Angles

An **angle** is the space between two lines that interesect.

Complementary Angles are two angles that add to 90 degrees.

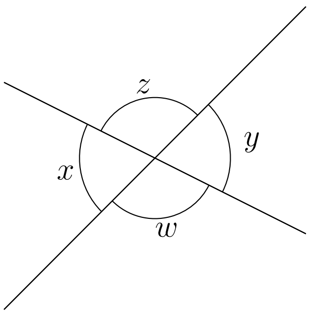
In the figure above we see how angles a and b form a right angle. Therefore we can say that angles a and b are complementary and thus: $a + b = 90°$

Supplementary Angles are two angles that add up to 180 degrees.

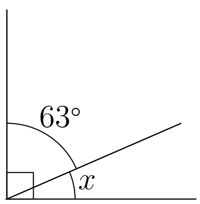
In the figure above we can see how angles a and b connect to form a semicircle and an angle of 180°. Therefore we can say that a and b are supplementary and thus $a + b = 180°$.

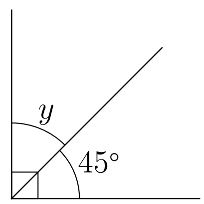
Opposite Angles

Opposite angles are the angles that are opposite each other when two lines intersect. Using what we know about supplementary angles, what property do opposite angles have?

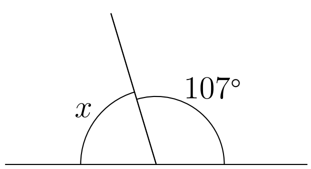


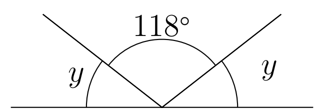
Example 1: Find the unknown complementary angle.

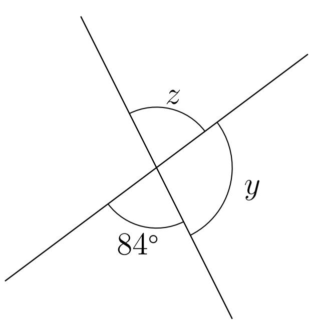
1. 

2. 

Example 2: Find the unknown angles.

1. 

2. 

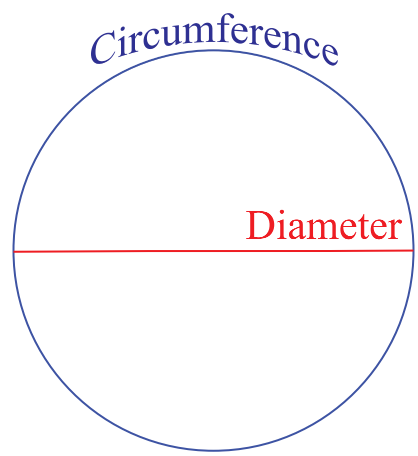
3. 

Introduction to Circles

1. **Measuring Circumference:** Lay the piece string around the circumference of the circle. Cut the string so it matches the Circumference in length. Measure the length of the cut string with a ruler. Record your measurement below.

2. **Measuring Diameter:** Use your ruler to measure across the red diameter of the circle. Record your measurement below.

3. **Comparing:** Find the ratio (quotient) between the circumference and diameter of your circle. Do this by dividing the circumference by the diameter.



Based on our formula for circumference of a circle what should we expect the ratio between our circumference and diameter to be?
Remember:

$$Circumference = \pi \times Diameter \rightarrow C = \pi d$$

From this equation we can see that circumference is equal to pi multiplied by d.

With the help of Algebra, we can solve for the ratio of Circumference and diameter! (divide by d on both sides)

$$\frac{C}{d} = \frac{\pi d}{d} \rightarrow \frac{C}{d} = \pi$$

From solving the above equation we see that the ratio between the circumference and the diameter is equal to pi. The very definition of pi is the ratio of a circles circumference and diameter. Thus, the ratio we found in our warm up should be approximately pi (around 3.) Mathematics is more than numbers themselves, it is understanding all that is behind them. There is no doubt that memorizing 67 digits of pi is impressive and should be shown off to friends, but also understanding the background of where this magic number ”pi” and where it came from is pretty neat. From now on instead of thinking of pi as just 3.14 or 22/7 we can additionally recognize that it is the ratio of circumference and diameter. Gaining understanding and making connections is key in Mathematics!

Knowing the above information lets do some thinking experiments...

What would we expect to happen to the ratio from our warm up exercise if we made the following modifications to our circle:

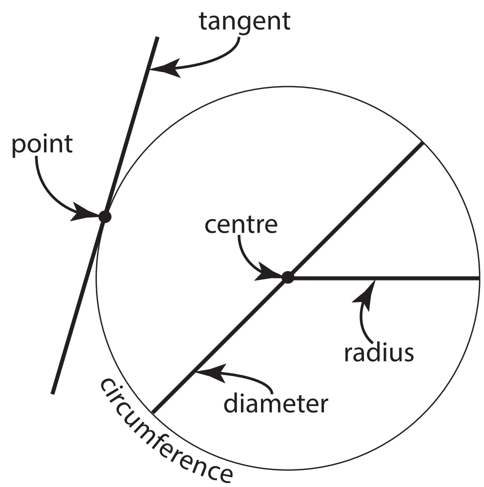
What if we increased the diamater?

What if we decreased it?

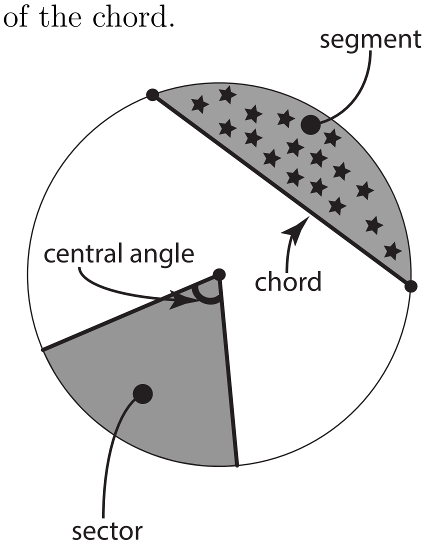
What if we decreased or increased the circumference?

Terminology

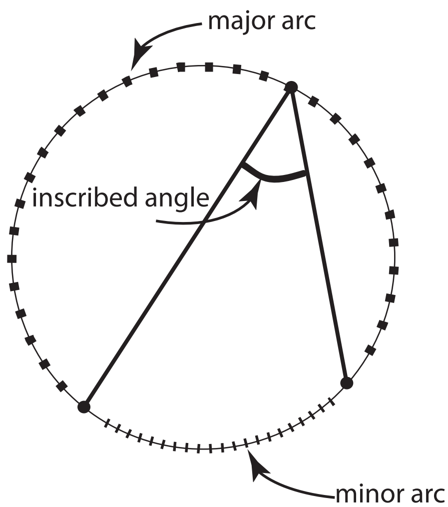
The diagram below depicts four terms you should already know: ***circumference***, ***centre***, ***radius***, and ***diameter***. There are also a couple of new concepts on this diagram. For our purposes, a ***point*** is any location on the circumference of a circle. A ***tangent*** is a line that passes through only one point on a circle's circumference.



A ***sector*** is a portion of a circle trapped by two radii (plural of radius). A ***central angle*** is an angle whose vertex is the centre of a circle and whose sides are radii intersecting the circle in two distinct points. We say the central angle is *subtended* by the ***arc*** (section of the circumference) between the two distinct points. A ***chord*** is a line segment that connects two distinct points of a circle. A ***segment*** is a portion of a circle made by a chord and an arc between the two endpoints of the chord.

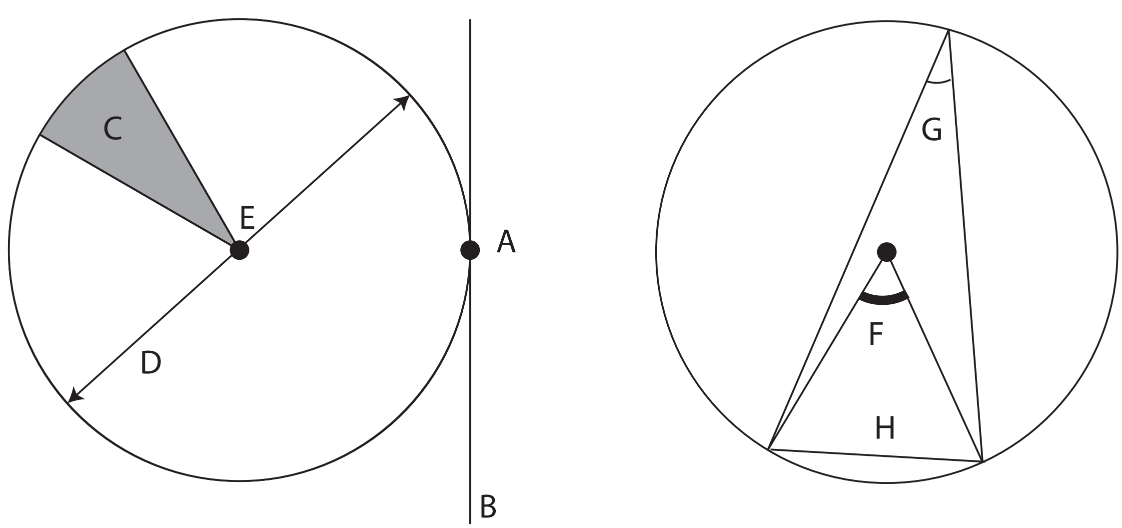


A ***major arc*** is the longer arc joining two points on the circumference of a circle. A ***minor arc*** is the shorter arc joining two points on the circumference of a circle. An ***inscribed angle*** is an angle formed by two chords in a circle which have a common endpoint.



Exercise:

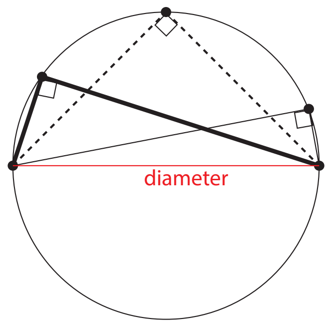
1. Label each letter with the correct name on the two circles below.



Circle Properties

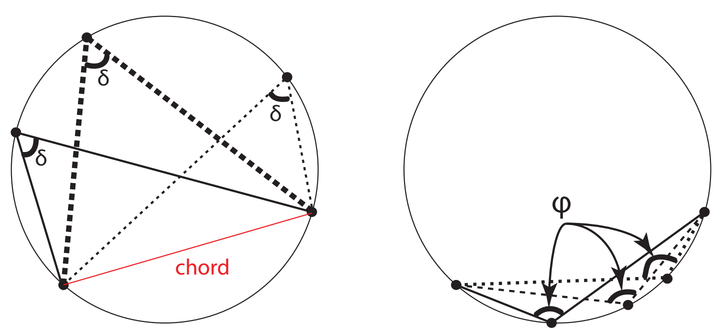
Angle in a Semicircle Theorem (AST):

An angle inscribed in a semicircle (i.e. the endpoints are at either end of the diameter) is always a right angle.



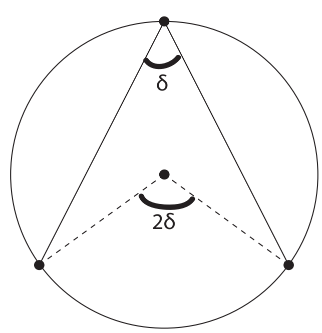
Angles Subtended by the Same Arc Theorem (ASAT):

An inscribed angle is always the same along the same arc where the endpoints are fixed.



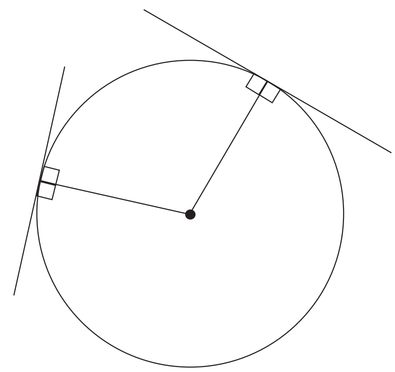
Central Angle Theorem/Star Trek Theorem (STT):

An inscribed angle is half of the corresponding central angle.



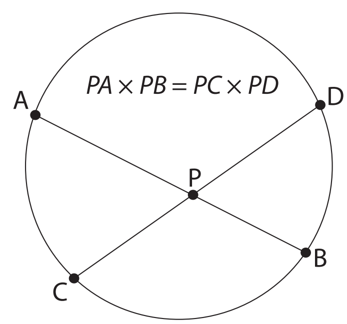
Tangent-Radius Theorem (TRT):

If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of tangency.



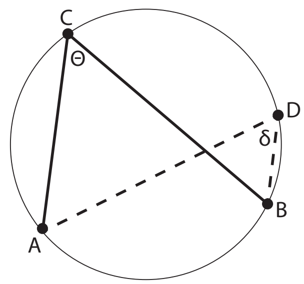
Crossed Chord Theorem (CCT):

If two chords intersect inside a circle then the product of the lengths of the segments of one chord equals the product of the lengths of the segments of the other chord.

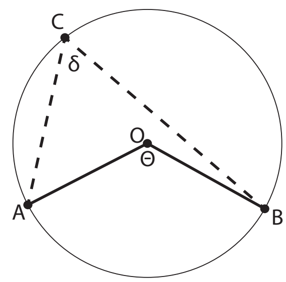


Practice:

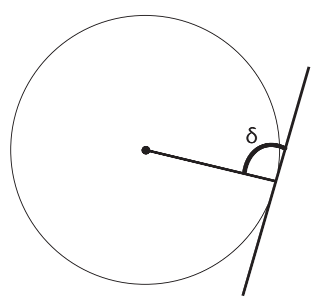
1. If the angle θ on the diagram below is 37°, what is the angle δ?



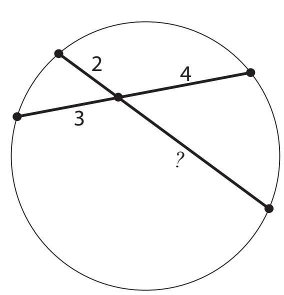
2. If the angle θ on the diagram below is 110°, what is the angle δ?



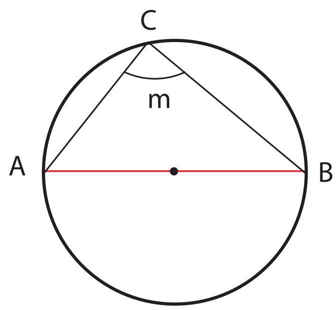
3. What is the angle δ in the diagram below?



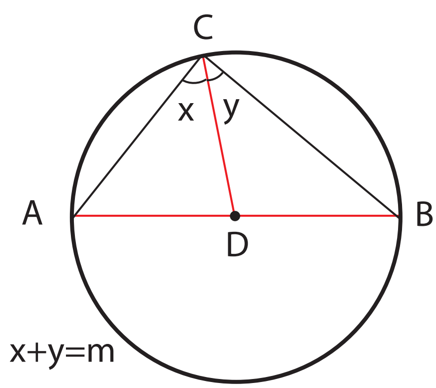
4. What is the missing line segment length in the diagram below?



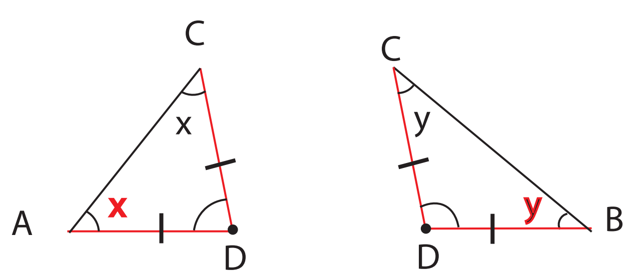
Proof: Angle in a Semicircle Theorem



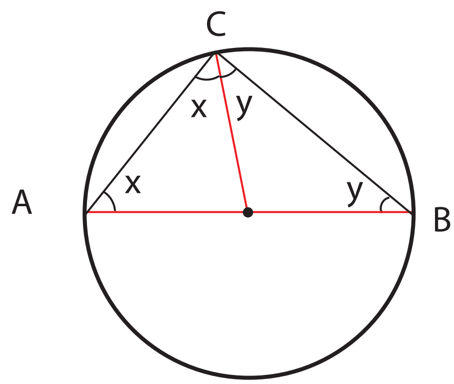
Don't take my word for it let's try to prove that this result is true and is not purely magic. **We want to show that angle m in the figure is always 90°.** We can first divide up the angle by drawing a line connecting point C to the center of the circle.



Labelling the center of the circle D, we now see we have two triangles $\triangle ACD$ and $\triangle BCD$. We can notice something very important about these triangles. **Line segments AD, CD and BD all run from the circumference of the circle to the center to make a radius.**



This means that AD, CD, and BD all are the same length thus $\triangle ACD$ and $\triangle BCD$ are isosceles triangles having 2 equal sides. Therefore these triangles must have 2 equal angles to match the 2 equal sides. $\angle CAD$ must be x and $\angle CDB$ must be y.



Let's look at the larger triangle ABC now. We know that the sum of its angles must be equal to 180°.

Mathematically this looks like:

$$\angle CAB + \angle ACB + \angle ABC = 180$$

$$\angle CAB = x, \ \angle ACB = x + y, \ \angle ABC = y$$

Subsituting in for the angles we know:

$$x + (x + y) + y = 180$$

$$2x + 2y = 180$$

Remember from the very beginning we established that m=x+y. Therefore we can say that 2x+2y would be double the value of m.

Back to our equation:

Since x+y=m

$$2x + 2y = 2m$$

$$2m = 180$$

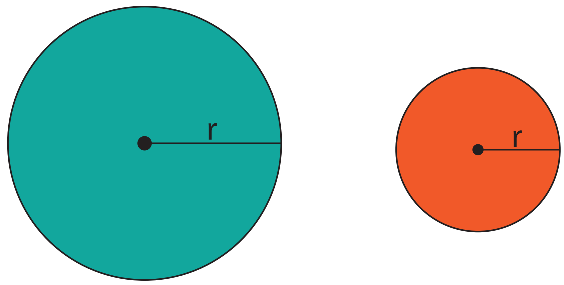
If double m is 180 then half of 180 must be equal to m.

$$\frac{2m}{2} = \frac{180°}{2}$$

$$m = 90°$$

Thus we have proved the result to be true. Any angle inscribed on the diamater is a right angle. As a challenge you can prove the other identities a similar way but require some algebra that is a little more advanced.

Area of Circles: Let's look at the area of circles. Logically, the area of a circle depends on its radius. The larger the radius the larger the circle will be.



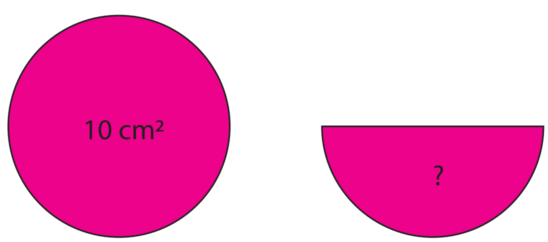
To be accurate the formula is $Area = \pi r^2$. or $Area = \pi(r \times r)$

Practice:

1. Find the area of a circle with radius 4cm. Try expressing your answer in terms of π.

2. What about the area of a circle with diamater 10cm?

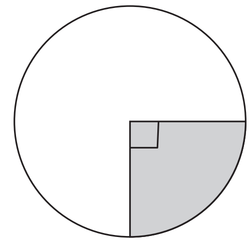
Area of a Sector: Given the following picture and told that the area of the circle on the left was $10cm^2$ what would you say that the area of the circle on the right is?



We can say that the semi-circle on the right represents a sector of a circle. As mentioned before a sector is a section of a circle trapped by two radii like a piece of pi. All sectors can be described as a fraction of a circles total area.

We can see that the shaded sector on the circle to the right takes up a quarter of the circle. Thus we can say that:

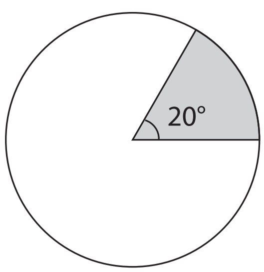
$$Area\ of\ Sector = Area\ of\ Circle \times \frac{1}{4}$$



In general we can say:

$$\boxed{Area\ of\ Sector = Area\ of\ Circle \times Fraction\ Shaded}$$

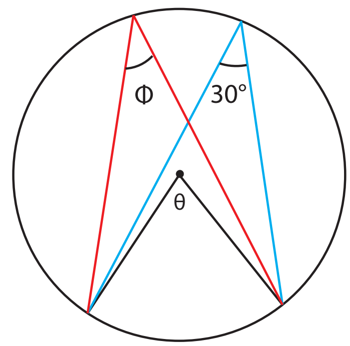
What if it is not that easy to see the fraction? How can we find the area of the following shaded sector if the circle has radius 6 cm? What about the perimeter of the sector?



PROBLEM SET:

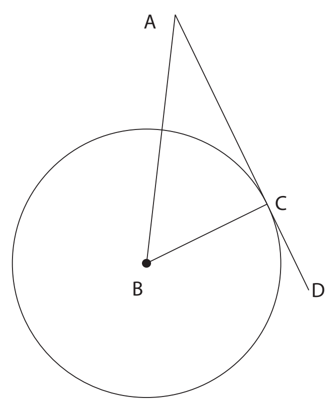
1. The circumference of a circle is 100 π cm. What is the radius of the circle?

2. Find the measure of angles θ and ϕ.

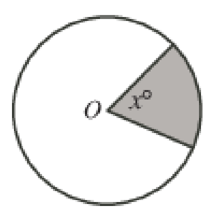


3. Line AD is tangent to the circle centered at point B.

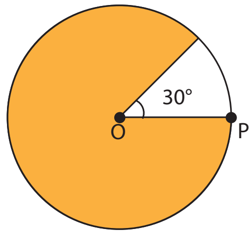
   Can we say that $\triangle ABC$ is a right triangle?



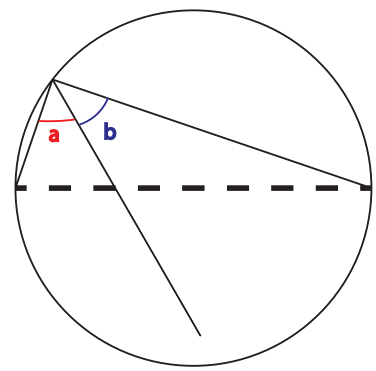
4. If 1/9 of the circle is shaded, what is the measure of x?



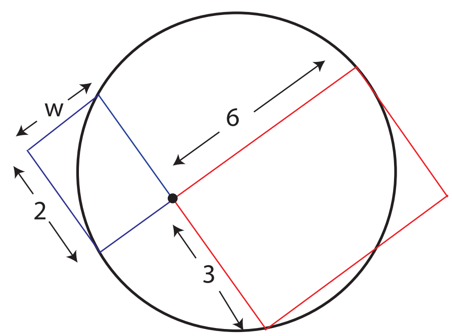
5. The length of OP is 3cm what is the area of the **white** section of the circle?



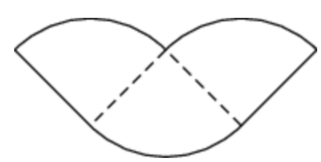
6. In the diagram below what is the relationship between angle a and angle b? If $b = 27°$ what is a?



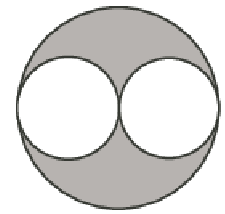
7. Find the areas of the two rectangles.



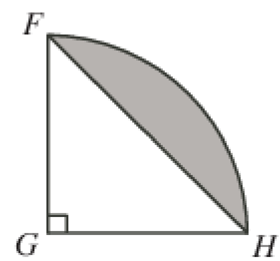
8. A circle with area 36π is cut into quarters and three of the pieces are arranged as shown. What is the perimeter and area of the resulting figure?



9. In the diagram, the smaller circles touch the larger circle and touch each other at the centre of the larger circle. The radius of the larger circle is 6. What is the area of the shaded region? (Source: 2010 Cayley (Grade 10), #13)



10. In right-angled, isosceles triangle FGH, FG=$\sqrt{8}$. Arc FH is part of the circumference of a circle with centre G and radius GH, as shown. What is the area of the shaded region?



11. * Three friends are in the park. Bob and Clarise are standing at the same spot and Abe is standing 10 m away. Bob chooses a random direction and walks in this direction until he is 10 m from Clarise. What is the probability that Bob is closer to Abe than Clarise is to Abe? (Source: 2014 Cayley (Grade 10), #23)