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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

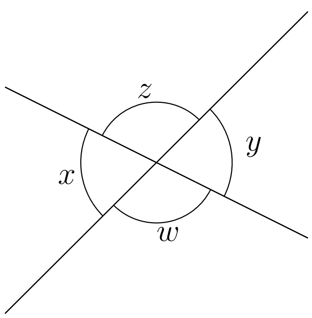
November 19/20

Circle Geometry Solutions

Angles

Opposite Angles

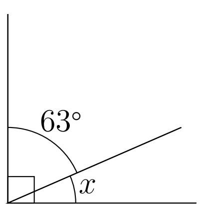
Opposite angles are the angles that are opposite each other when two lines intersect. Using what we know about supplementary angles, what property do opposite angles have?



Since w and y are supplementary angles and w and x are supplementary angles, we get that x is equal to y. By similar logic, z is equal to w. Therefore, opposite angles are equal.

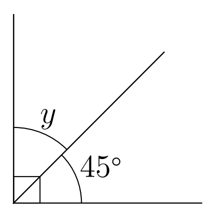
Example 1: Find the unknown complementary angle.

1.



$x = 90° - 63° = 27°$

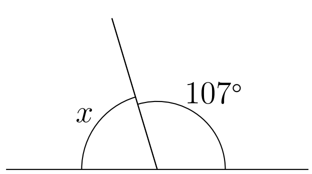
2.



$y = 90° - 45° = 45°$

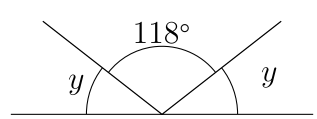
Example 2: Find the unknown angles.

1.



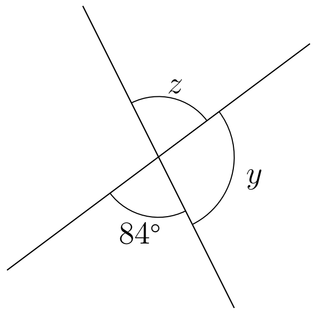
$x = 180° - 107° = 73°$

2.



$118° + 2y = 180°$ so we get $2y = 62°$. Thus, $y = 31°$

3.



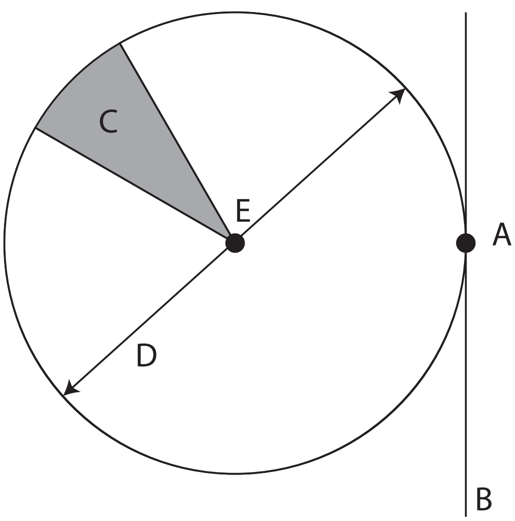
$y = 180° - 84° = 96°$ by supplementary angles. z is the opposite angle to 84° so $z = 84°$

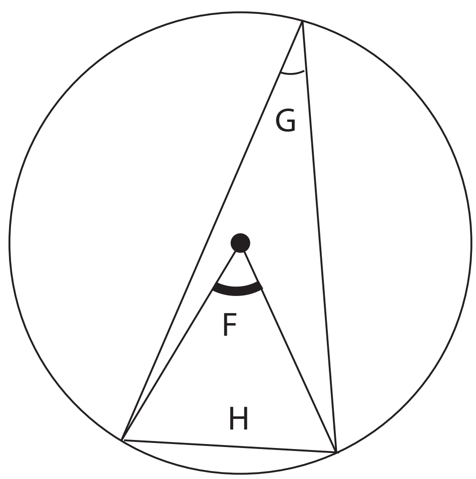
Terminology

Exercise:

1. Label each letter with the correct name on the two circles below.

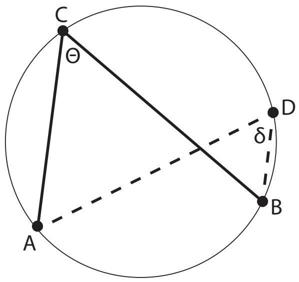
A: point, **B:** tangent, **C:** Sector, **D:** Diameter, **E:** Center, **F:** Central Angle, **G:** Inscribed Angle, **H:** Chord.





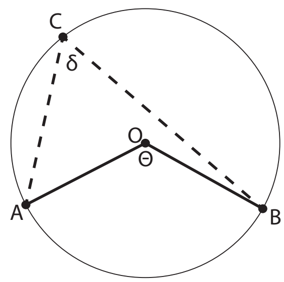
Circle Properties Practice:

1. If the angle θ on the diagram below is $37°$, what is the angle δ? What is the angle ϕ?



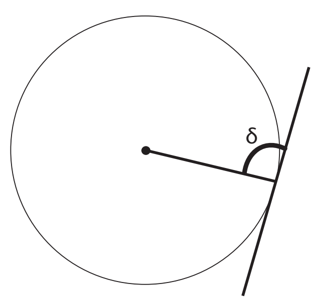
By ASAT $\delta = \theta = 37°$ since they are inscribed angles on the same chord.

2. If the angle θ on the diagram below is $110°$, what is the angle δ?



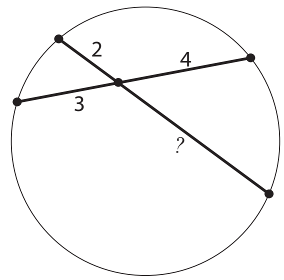
By STT the central angle θ is double the inscribed angle δ. Thus $\delta = 110 \div 2 = 55$

3. What is the angle δ in the diagram below?



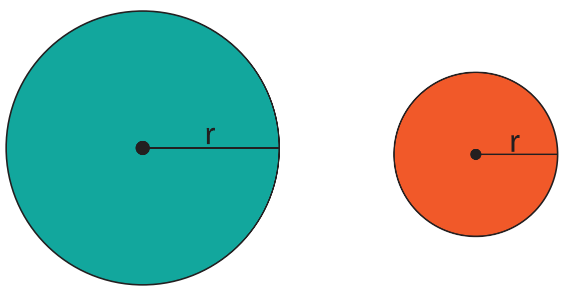
We have angle δ between a tangent line (only touches one point on the circle) and a radius. By TRT we can say the lines are perpendicular so $\delta = 90°$

4. What is the missing line segment length in the diagram below?



By CCT we say that $4 \times 3 = 2 \times ?$. We can rearrange this to find the missing side: If $2 \times ?$ is equal to 12 then ? must be 6.

Area of Circles: Let's look at the area of circles. Logically, the area of a circle depends on its radius. The larger the radius the larger the circle will be.



To be accurate the formula is $Area = \pi \times r^2$. or $Area = \pi(r \times r)$

Practice:

1. Find the area of a circle with radius 4cm. Try expressing your answer in terms of π.

$Area = \pi r^2$, r=4cm

$Area = \pi \times (4cm)^2$

$Area = 16\pi cm^2 \approx 50.27cm^2$ If we wanted to express area in terms of π we would stop at $Area = 16\pi cm^2$

2. What about the area of a circle with diameter 10cm? Radius is half the length of diameter.

$Radius = 10 \div 2 = 5$

$Area = \pi r^2$

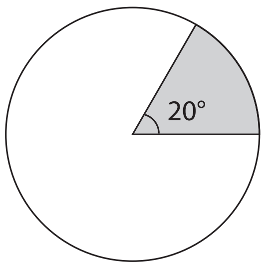
$Area = \pi \times (5cm)^2$

$Area = 25\pi \approx 78.54cm^2$

Area of a Sector:

What if it is not that easy to see the fraction? How can we find the area of the following shaded sector if the circle has radius 6 cm? What about the perimeter of the sector?

We can find the fraction of the circle that is shaded by comparing the central angle of the sector to the total degrees in a circle.



$$Fraction\ Shaded = \frac{Sector\ Angle}{Degrees\ in\ Circle}$$

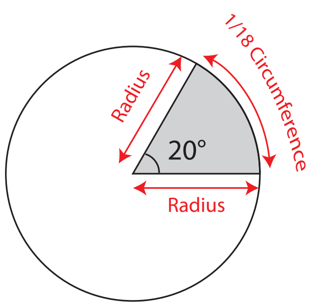
$$Fraction\ Shaded = \frac{20°}{360°} = \frac{1}{18}$$

Area of Sector = Area of Circle × Fraction Shaded = $(\pi 6^2) \times \frac{1}{18}$

$Area\ of\ Sector = 36\pi \times \frac{1}{18}$

$Area\ of\ Sector = 2\pi \approx 6.28cm.$

Finding Perimeter:



The arc at the edge of the sector can be said to be 1/18 of the circumference as we established that the sector central angle is 1/18 of the total circle.

The two edges in the circle are both radii as they run from the center to the edge of the circle. Thus we can say that:

$Perimeter = Radius + Radius + \frac{1}{18} \times Circumference$

$Perimeter = 2 \times Radius + \frac{1}{18} \times 2\pi r$

$Perimeter = 2 \times 6 + \frac{1}{18} \times 2 \times \pi \times 6$

$Perimeter = 12 + \frac{2 \times 6}{18}\pi$

$Perimeter = 12 + \frac{2}{3}\pi$

$Perimeter \approx 25.13cm$

PROBLEM SET:

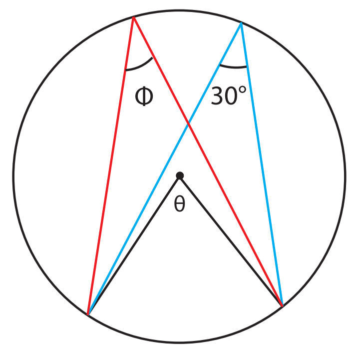
1. The circumference of a circle is 100 π cm. What is the radius of the circle?

   Circumference $= \pi \times diamater = \pi \times 100$

   By inspecting the two equations we can see that diamater must be 100. Knowing that radius is half of the diamater of a circle we can find the radius: $Radius = 100 \div 2 = 50$

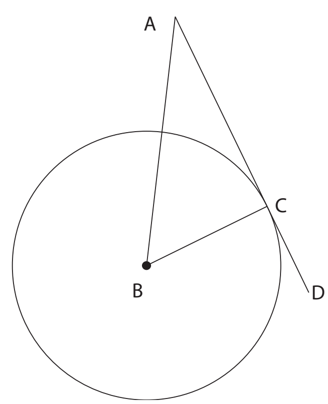
2. Find the measure of angles θ and ϕ.

   By ASAT we can say that angles ϕ and 30° must be the same as they are inscribed angles on the same chord.Thus $\phi = 30°$ By STT we can say that angle θ is twice as large as ϕ and 30°. Thus $\theta = 2 \times 30° = 60°$.



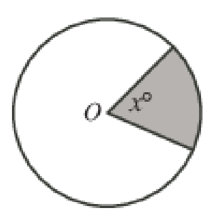
3. Line AD is tangent to the circle centered at point B. Can we say that $\triangle ABC$ is a right triangle?

   BC is by definition a radius as it runs from the centre of the circle to the edge. AD is tangent to the circle and also parallel to AC. Thus AC is a tangent line also. By TRT we know that a tangent line and radius are perpindicular when they meet. Thus we can say that $\angle BCA = 90°$ and thus $\triangle ABC$ is a right triangle.

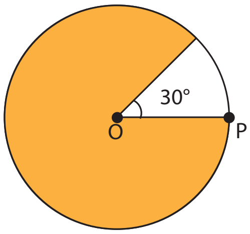


4. If 1/9 of the circle is shaded, what is the measure of x?

   There are in total 360° in a circle. If $\frac{1}{9}$ of the circle is shaded then our central angle x should be one ninth of that total 360°. $\angle x = \frac{1}{9} \times 360° = 40°$



5. The length of OP is 3cm what is the area of the **white** section of the circle?



The shaded sector on the circle has a central angle of $360° - 30° = 330°$

We can find the fraction that is shaded by comparing 330° to the total degrees in the circle 360°.

$Fraction\ White = \frac{30°}{360°} = \frac{1}{12}$

The area of the full circle can be found by $area = \pi r^2$, where r is the radius of the circle.

Since length OP runs from the center to the circumference of the circle we can call it a radius. $OP = Radius = 3cm$.

Thus $Area = \pi \times 3cm^2$

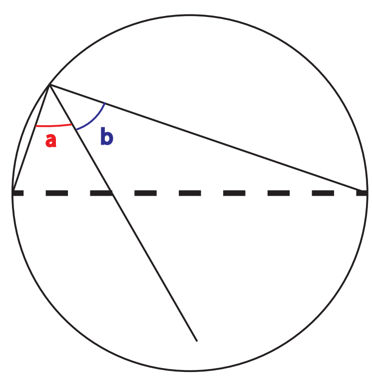
$Area = 9\pi$

$Area\ Sector = Area\ total\ Circle \times Fraction\ White$

$Area\ Sector = 9\pi \times \frac{1}{12}$

$Area\ Sector = \frac{9}{12}\pi = \frac{3}{4}\pi \approx 2.36cm^2$

6. In the diagram below what is the relationship between angle a and angle b? If $b = 27°$ what is a?



By ASAT angles a and b add up to 90° because they are inscribed on a diameter. Thus we can say the angles are complementary.

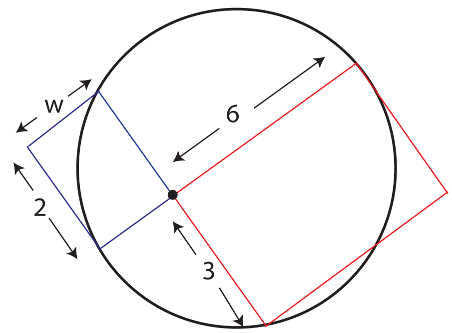
Mathematically $\angle a + \angle b = 90°$, also we know $b = 27°$

$\angle a + \angle 27° = 90°$

$\angle a = \angle 90° - \angle 27°$

$\angle a = \angle 63°$

7. Find the areas of the two rectangles.



The sides of the rectangles inside the circle are crossing chords. Thus we can say by CCT:

$6 \times w = 3 \times 2$

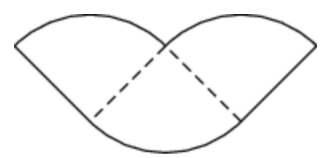
$6w = 6$

$w = 1$

Area of smaller rectangle $= 2 \times 1 = 2$

Area of larger rectangle $= 6 \times 3 = 18$

8. A circle with area 36π is cut into quarters and three of the pieces are arranged as shown. What is the perimeter and area of the resulting figure?



To find the Area:

We have 3 quarter circles pieced together in the figure shown.

Thus we have $\frac{3}{4}$ of the original area of the circle left.

$Area\ of\ Figure = Area\ of\ Full\ Circle \times \frac{3}{4}$

$Area\ of\ Figure = 36\pi \times \frac{3}{4}$

$Area\ of\ Figure = 27\pi$

To Find the Perimeter:

We have 5 discrete sides to the perimeter. We have 3 quarter arcs and 2 radii.

The quarter arcs are each a quarter of the original circles circumference.

Thus $Perimeter = \frac{3}{4} \times circumference + 2 \times Radius$

We must find the radius of the original circle:

We know that Area $= \pi \times r^2 = 36\pi$

The radius must be a number than when squared is 36: $r^2 = 36$

The radius that satisifies this condition is r=6: $6^2 = 36$

Finding Circumference:

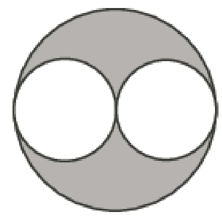
$Circumference = \pi \times d = 2\pi r$

$Circumference = 2\pi \times 6$

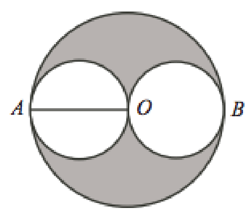
$Circumference = 12\pi$

$Perimeter = \frac{3}{4} \times 12\pi + 2 \times 6$ $Perimeter = 9\pi + 12 \approx 49.7$

9. In the diagram, the smaller circles touch the larger circle and touch each other at the centre of the larger circle. The radius of the larger circle is 6. What is the area of the shaded region? (Source: 2010 Cayley (Grade 10), #13)



Label the centre of the larger circle O and the points of contact between the larger circle and the smaller circles A and$B$. Draw the radius OA of the larger circle.



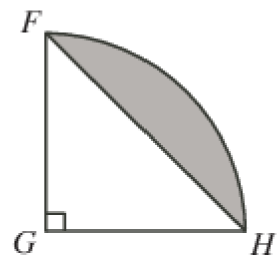
Since the smaller circle and the larger circle touch at A, then the diameter through A of the smaller circle lies along the diameter through A of the larger circle. (This is because each diameter is perpendicular to the common tangent at the point of contact.) Since AO is a radius of the larger circle, then it is a diameter of the smaller circle. Since the radius of the larger circle is 6, then the diameter of the smaller circle is 6, so the radius of the smaller circle on the left is 3.

Similarly, we can draw a radius through O and B and deduce that the radius of the smaller circle on the right is also 3. The area of the shaded region equals the area of the larger circle minus the combined area of the two smaller circles. Thus, the area of the shaded region is:

Area Shaded = Area Larger Circle - Area small circle A - Area small circle B

Area Shaded = $\pi \times 6^2 - \pi \times 3^2 - \pi \times 3^2 = 36\pi - 9\pi - 9\pi = 18\pi$ Area Shaded = $18\pi \approx 56.55$

10. In right-angled, isosceles triangle FGH, FG=$\sqrt{8}$. Arc FH is part of the circumference of a circle with centre G and radius GH, as shown. What is the area of the shaded region?



Arc FH is a quarter circumference of a circle centered at G. Thus we can say the circle has radius $FG = GH = \sqrt{8}$ Notice how the figure shows a quarter with a triangle FGH cut out of it.

Thus to find the shaded area we can find the difference of the areas of the quarter circle and $\triangle FGH$.

$Shaded\ Area = Area\ Quarter\ Circle - Area \triangle FGH$

$Area\ of \triangle FGH = \dfrac{base \times height}{2} = \dfrac{\sqrt{8} \times \sqrt{8}}{2} = \dfrac{8}{2} = 4$

$Area\ Quarter\ Circle = Area\ Circle \times \dfrac{1}{4}$

$Area\ Quarter\ Circle = \pi(\sqrt{8})^2 \times \frac{1}{4} = 8\pi \times \frac{1}{4} = 2\pi$

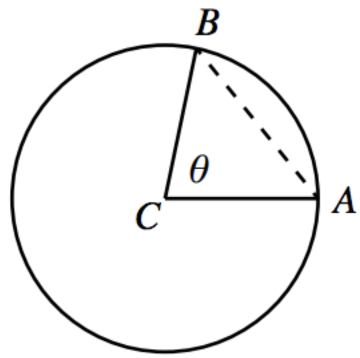
$Shaded\ Area = 8\pi - 4 \approx 21.13$

11. * Three friends are in the park. Bob and Clarise are standing at the same spot and Abe is standing 10 m away. Bob chooses a random direction and walks in this direction until he is 10 m from Clarise. What is the probability that Bob is closer to Abe than Clarise is to Abe? (Source: 2014 Cayley (Grade 10), #23)

We call Clarises spot C and Abes spot A.

Consider a circle centred at C with radius 10 m. Since A is 10 m from C, then A is on this circle.

Bob starts at C and picks a direction to walk, with every direction being equally likely to be chosen. We model this by having Bob choose an angle θ between $0°$ and $360°$ and walk 10 m along a segment that makes this angle when measured counterclockwise from CA. Bob ends at point B, which is also on the circle.



We need to determine the probability that AB < AC.

Since the circle is symmetric above and below the diameter implied by CA, we can assume that θ is between 0° and 180° as the probability will be the same below the diameter.

Consider $\triangle CAB$ and note that $CA = CB = 10$ m.

It will be true that $AB < AC$ whenever AB is the shortest side of $\triangle ABC$.

AB will be the shortest side of $\triangle ABC$ whenever it is opposite the smallest angle of $\triangle ABC$.(In any triangle, the shortest side is opposite the smallest angle and the longest side is opposite the largest angle.)

Since $\triangle ABC$ is isosceles with $CA = CB$, then $\angle CAB = \angle CBA$.

We know that $\theta = \angle ACB$ is opposite AB and $\angle ACB + \angle CAB + \angle CBA = 180^\circ$

Since $\angle CAB = \angle CBA$, then $\angle ACB + 2 \triangle CAB = 180^\circ$ or $\angle CAB = 90^\circ - \frac{1}{2}\angle ACB$.

If $\theta = \angle ACB$ is smaller than 60°, then $\angle CAB = 90^\circ - \frac{1}{2}\theta$ will be greater than 60°.

Similarly, if $\angle ACB$ is greater than 60°, then $\angle CAB = 90^\circ - \frac{1}{2}\theta$ will be smaller than 60°.

Therefore, AB is the shortest side of $\triangle ABC$ whenever θ is between 0° and 60°.

Since θ is uniformly chosen in the range 0° to 180° and $60^\circ = \frac{1}{3} \times 180^\circ$, then the probability that θ is in the desired range is $\frac{1}{3}$.

Therefore, the probability that Bob is closer to Abe than Clarise is to Abe is $\frac{1}{3}$

(Note that we can ignore the cases $\theta = 0^\circ$, $\theta = 60^\circ$ and $\theta = 180^\circ$ because these are only three specific cases out of an infinite number of possible values for θ.)