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Grade 6 Math Circles

Fall $[2 \times (3 \times 4 + 1000) - 5]$ - Oct 15/16

Algebra

Algebra is the part of mathematics in which letters and other general symbols are used to represent numbers and quantities in formulae and equations. Algebra is useful in areas like:

Geometry, Computer Programming, Finances, Sales, Construction

Variable - a variable is a symbol used for a number we don't know yet. Usually variables are represented by cursive letters, commonly x and y, but any other letter is appropriate as well.

Constant - a number on its own.

Equation - a mathematical statement that 2 expressions on each side of an equal sign are equal.

Exercise: *For the following equations, state the variable(s) and the constant(s).*

1. $13 = 8 + 5$ Constant(s): __________ Variable(s): __________
2. $a + 27 = 49$ Constant(s): __________ Variable(s): __________
3. $36 - 2z = 18$ Constant(s): __________ Variable(s): __________

Solving Equations - to solve an equation is to find what number the given variable represents to maintain equality. Solving equations can be useful in areas like:

- **Finances**: $[200(1 - x) = 95]$ where x is an interest rate.
- **Sales**: $17p = \$68$ where p is the price of one T-shirt.
- **Construction**: $13w = 104$ where w is the width of the pool.

Steps to Solving an Equation

Before we learn the steps, we need to learn the following term and operation.

Collecting Like-Terms

Like-Terms: terms whose variables are the same.

Examples: *Constants such as 13 and 78 are like-terms as they have no variable.* x, $3x$, $\frac{5}{7}x$ *are like-terms. 4 and* $4x$ *are* **not** *like-terms as one has a variable but the other doesn't.*

Collecting like-terms is to perform the operations between like-terms.

Example: *In the equation* $x + 9 + 2 \times 4 + 2x = 35 \div 7 + 88x \div 11x$*, the like terms on each side can be collected to produce the following simplified equation:* $3x + 17 = 5 + 8x$.

We can sometimes guess the variable (*e.g.* when given $3 \times y = 12$, it can easily be seen that $y = 4$) but guessing doesn't always work so we need a reliable method to determine the value of the variable in any equation.

The goal is to get the variable by itself on one side of the equation using the following steps

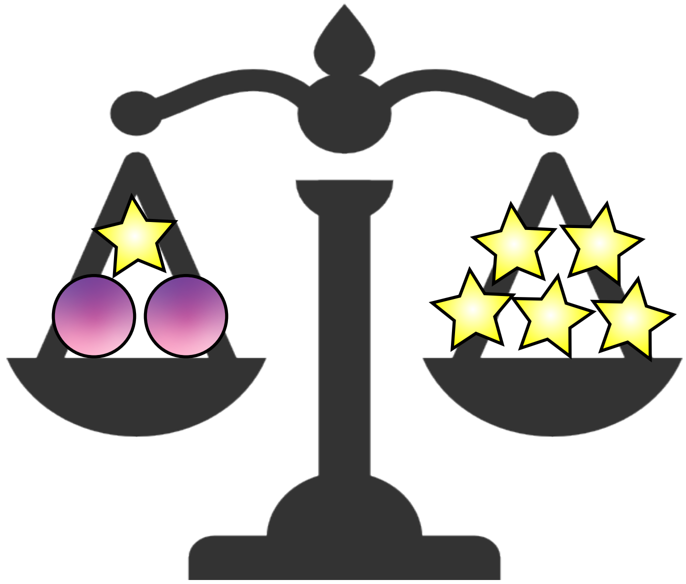
1. Determine the variable in the equation (*i.e.* what you are trying to solve for).
2. If possible, simplify each side of the equation by **collecting like-terms**.
3. Isolate the variable by performing **opposite operations** to eliminate the constants. Don't forget to collect like-terms after performing each opposite operations!

An Important Rule:

What you do to one side of the equation, you **must** *do to the other side of the equation.*

Initial Operation	Opposite Operation
Addition	Subtraction
Subtraction	Addition
Multiplication	Division
Division	Multiplication

Consider the even balance scale shown below. Each ☆ weighs 1 kg, how much does each ◯ weigh?



This same situation can be expressed in terms of numbers and variables. Since each ☆ weighs 1 kg, and the balance is even (each side is equal) we can write the following equation and solve with the steps given above:

- We begin with our equation:

$$2\bigcirc + 1 = 5$$

- We now subtract 1 from both sides giving us:

$$\begin{aligned} 2\bigcirc + 1 - 1 &= 5 - 1 \\ 2\bigcirc &= 4 \end{aligned}$$

- Now dividing both sides by the constant in front of the ◯ we are left with:

$$\begin{aligned} 2\bigcirc \div 2 &= 4 \div 2 \\ \bigcirc &= 2 \end{aligned}$$

Which is what we already knew!

Try It At Home!

http://www.mathsisfun.com/algebra/add-subtract-balance.html

Exercise Set 1:

Determine the value of the variable x in each of the following equations. Challenge questions *.

(*Remember:* We always want to solve for x rather than $-x$):

a)
$$\begin{aligned} x+3 &= 9 \\ x+3-3 &= 9-3 \\ x &= 6 \end{aligned}$$

b)
$$\begin{aligned} 5-x &= 4 \\ 5-x+x &= 4+x \\ 5-4 &= 4+x-4 \\ 1 &= x \end{aligned}$$

c)
$$\begin{aligned} 2x &= 6 \\ 2x\div 2 &= 6\div 2 \\ x &= 3 \end{aligned}$$

d) $5+x=12$

e) $-x+6=5$

f) $7+3x=13$

g) $x+3+4+x=10-1$

h) $5+3=x+8$

i) $-8+1+4x=11-x$

* j) $\frac{1}{2}x-3=1$

* k) $\frac{2}{9}x-4=x-1$

** l) $\frac{5}{x}+2=\frac{7}{2}$

Properties of Algebra

Sometimes, a given equation is very long and contains more than one operation. The following properties and rules in Algebra can help us solve these longer equations:

Order of Operation

The order in which we calculate each element of the equation is determined by the ***order of operation****. The acronym that can help us follow the steps to ensure we get the right answer is called BEDMAS*:

Brackets	First Priority
Exponents	Second Priority
Division	Third Priority
Multiplication	Third Priority
Addition	Fourth Priority
Subtraction	Fourth Priority

Priorities mean that we evaluate brackets before exponents, evaluate exponents before we multiply, divide before we subtract, etc.

Note: *If an equation has two or more operations of the same priority, do those operations from left to right.*

Exercise: *Evaluate.*

$10 + 2 \div 2 - 3 \times 3 =$ __________________ $6 \times (5 + 2 \times 6 \div 6 + 8) \div 10 =$ __________________

Commutative Property (CP)

CP of Multiplication: *the order in which we multiply numbers does not change the product.*

$$a \times b = b \times a$$

CP of Addition: *the order in which we add numbers does not change the sum.*

$$a + b = b + a$$

Exercise: *Fill in the blanks.*

$8\times2 = 2\times$____ $3\times y =$ ____$\times3$ $4+5 =$ ____$+4$ $7+x = x+$____

Associative Property (AP)

AP of Multiplication: *the way in which numbers are grouped in multiplication does not change the product.*

$$(a \times b) \times c = a \times (b \times c)$$

AP of Addition: *the way in which numbers are grouped in addition does not change the sum.*

$$(a + b) + c = a + (b + c)$$

Exercise: *Which expressions are equivalent to* $(9 \times 2) \times 5$*?*

$5 \times (9 \times 2)$ $\quad$ 11×5 $\quad$ $(2 \times 5) \times 9$ $\quad$ 9×10 $\quad$ $9 \times 2 \times 5$

Distributive Property (DP)

*The **distributive property of multiplication** tells us how to solve expressions in the form of:*

$$a \times (b + c) = (a \times b) + (a \times c)$$
$$a \times (b - c) = (a \times b) - (a \times c)$$

Note: *This property is also sometimes called the **distributive law of multiplication and division**.*

Exercise: *Expand and simplify the following expressions using the DP and collecting like terms.*

$4 \times (5 + 9) =$ ______________________ $\quad$ $3 \times (x + 2) =$ ______________________

$5 \times (4x - 7) =$ ____________________ $\quad$ $8 \times (3x + 2 - 9) =$ __________________

$* \ \frac{1}{2} \times (400 + 68) =$ __________________ $\quad$ $* \ \frac{1}{5} \times (125 + x) =$ __________________

Exercise Set 2:

Determine the value of the variable x in each of the following equations. Challenge questions *.

(*Remember:* We always want to solve for x rather than $-x$):

a)
$$\begin{aligned} 4\times(x+2) &= 20 \\ (4\times x)+(4\times 2) &= 20 \\ 4x+8-8 &= 20-8 \\ 4x\div 4 &= 12\div 4 \\ x &= 3 \end{aligned}$$

b)
$$\begin{aligned} 3\times(5\times y) &= 45 \\ (3\times 5)\times y &= 45 \\ 15\times y\div 15 &= 45\div 15 \\ y &= 3 \end{aligned}$$

c)
$$\begin{aligned} (6\times 2+7)\times z &= 57 \\ (12+7)\times z &= 57 \\ 19\times z\div 19 &= 57\div 19 \\ z &= 3 \end{aligned}$$

d) $(6+x)+4=17$

e) $(5-2x)-2=1$

f) $x-[3\times(2+4)]=2$

g) $x+3\times 4+x=(40-6)\div 2$

h) $\frac{1}{4}\times(52+x)=16$

i) $[14+(24\div x)]\times 4=68$

* j) $x+(-4)-8\times[2\div((-10)\div 10)]=10$

* k) $\frac{1}{3}\times(84-7x)=1$

Solving for Two Variables:

In many situations, we have to solve for more than one unknown variable. There is many methods to doing this but we will only look at one.

Substitution is a technique for solving systems of linear equations.

Steps:

Suppose we are given two equations with two variables x and y.

1. Solve one of the equations for one variable *(suppose we solve for $x = ...$)*.
2. Substitute the expression $x = ...$ into the other equation to solve for the other variable, y.
3. Substitute your answer into the first equation (if needed) to solve for x.
4. Check your solution.

Example: Solve for both variables in the following examples.

a) 1. $n + 4 = 9$

2. $n + 4 + t = 17$

Using Equation 1:

$$\begin{aligned} n + 4 - 4 &= 9 - 4 \\ n &= 5 \end{aligned}$$

Substituting into Equation 2:

$$\begin{aligned} 5 + 4 + t &= 17 \\ 9 + t - 9 &= 17 - 9 \\ t &= 8 \end{aligned}$$

b) 1. $x \div 4 = 2$

2. $x + y = 13$

Using Equation 1:

$$\begin{aligned} x \div 4 \times 4 &= 2 \times 4 \\ x &= 8 \end{aligned}$$

Substituting into Equation 2:

$$\begin{aligned} 8 + y &= 13 \\ 8 + y - 8 &= 13 - 8 \\ y &= 5 \end{aligned}$$

Step (4) to check your solution is explained below.

Check your Solution

Once we have found the value of our variables, we must check if they are correct and satisfy the equations. To do so, we check if the **left hand side (LHS)** of the equation is equal to the **right hand side (RHS)**.

a)
1. $n + 4 = 9$
2. $n + 4 + t = 17$

Checking Equation 1:

$$\begin{aligned} LHS &= 5 + 4 \\ LHS &= 9 \\ LHS &= RHS \end{aligned}$$

Since LHS = RHS, then $n = 5$.

Checking Equation 2:

$$\begin{aligned} LHS &= 5 + 4 + 8 \\ LHS &= 17 \\ LHS &= RHS \end{aligned}$$

Since LHS = RHS, then $t = 8$.

b)
1. $x \div 4 = 2$
2. $x + y = 13$

Checking Equation 1:

$$\begin{aligned} LHS &= 8 \div 4 \\ LHS &= 2 \\ LHS &= RHS \end{aligned}$$

Since LHS = RHS, then $x = 8$.

Checking Equation 2:

$$\begin{aligned} LHS &= 8 + 5 \\ LHS &= 13 \\ LHS &= RHS \end{aligned}$$

Since LHS = RHS, then $y = 5$.

Exercise Set 3:

Solve for both variables in the following examples and check that your solutions satisfy the equations. Challenge questions *.

c)
1. $s - 7 = r$
2. $8 - r = 3$

d)
1. $a \times 3 = 7 - 4$
2. $b + 2 = 9 - a$

e) *
1. $3x + y = -3$
2. $x = -y + 3$

f)*
1. $7x + 10y = 36$
2. $-2x + y = 9$

Algebra in Real Life

Word Problems with Algebra

Often we are given a word problem where we are given some information and must solve to determine some values. This can easily be done using the steps outlined below:

1. Identify what you are solving for and introduce the variables using let statements.
2. Write out the mathematical equations using your variables.
3. Solve the equation.
4. Write a conclusion.

Example - Video Games:

Two friends Nicolas and Austin are talking about their video game collection. Nicolas tells Austin: "if you buy thirteen more video games, you will have 10 video games more than me" In return, Austin tells Nicolas: "if you double the number of your games and give 1 of them away, you will have triple the number of games I have." Which friend has more games and how many more games do they have?

1. *Identify what you are solving for and introduce the variables using let statements.*

   We are solving for the number of video games each friend has to determine which friend has more video games and how many more games they have.
   Let N be the number of video games Nicolas has.
   Let A be the number of video games Austin has.

2. *Write out the mathematical equations using your variables.*

   Equation 1: $A + 13 = N + 10$
   Equation 2: $2N - 1 = 3A$

3. *Solve the equation.*

   We first use equation 1 to find an expression for N.

$$
\begin{aligned}
A + 13 &= N + 10 \\
A + 13 - 10 &= N + 10 - 10 \\
A + 3 &= N
\end{aligned}
$$

   Substitute this expression for N into equation 2 to get:

$$
\begin{aligned}
2(A+3) - 1 &= 3A \\
2A + 6 - 1 &= 3A \\
2A + 5 - 2A &= 3A - 2A \\
5 &= A
\end{aligned}
$$

   Substituting this back into the expression for N we get N = 8 so Nicolas has 8 video games and Austin has 5 video games.

4. *Write a conclusion.*

   Therefore, Nicolas has 3 video games more than Austin.

Bonus: Using Algebra in Magic

Algebra is used alot more often than you may think. Follow these steps and we will use algebra to read your mind (or try it on your friends!)

1. Write down the number of the month you were born in
2. Add 32 to that number
3. Add the difference between 12 and your birth month number (*eg:* If you are born in Jan. the difference is $12 - 1 = 11$)
4. Divide your new number by 2
5. Add 3 to that number
6. Find your special colour by using the following code:
   - If a=1, b=2, etc. Find the corresponding letter from the number you got in Step 5.
   - Using this letter, choose a colour that begins with the same letter

Ready for the magic? Is your colour **YELLOW**?

Figuring out the Magician's Secret:

Let's call your birth number x. If we follow all the steps we get an expression that looks as follows:

Using your knowledge can you figure out the magician's secret?

Hint: Look back at the second point in the Steps to Solving an Equation on Page 2

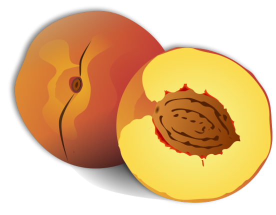
Problem Set:

1. Amy and her brother Derek are always fighting over who has the most toys. Their mother decides that the best way to solve the problem is to ensure both Amy and Derek have an equal number of toys. She finds out that Amy has 4 less toys than Derek. She also knows that Derek has 10 toys.

   (a) How many toys did Amy start with?

   (b) How many toys will they each have once all the toys are distributed evenly?

2. If 18 people share a basket of peaches evenly, each person gets 12 peaches. If there had been 6 fewer people, how many peaches would each person have gotten?

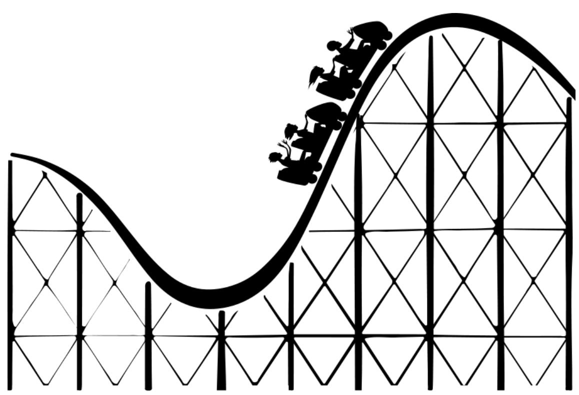


3. George has 9 nickles, 3 dimes, 7 quarters and 11 loonies in his pocket. What is the total amount of money that he has in his pocket?

4. The Hulk is 3 cm taller than Tarzan and 4 cm shorter than Superman. If Superman's height is 2 meters, how tall is Tarzan?

5. A classroom of 53 students is divided into two groups, with one of the groups having 7 students more than the other. What is the size of each group?

6. Derek decides to go to Wonderland for a day. He spends \$50 to get into the park and then spends 1/2 of his remaining money on food and games. He comes home with \$20. How much money did Derek start with?

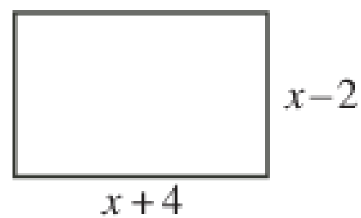


7. Zack is 30 years old. Cody is 2 years old. How many years will it take until Zack is only 3 times as old as Cody?

8. Each chef at Sushi Emperor prepares 15 regular rolls and 20 vegetable rolls. On Tuesday, each customer ate 2 regular rolls and 3 vegetable rolls. By the end of the day, 4 regular rolls and 1 vegetable roll remained uneaten. How many chefs and how many customers were in Sushi Emperor on Tuesday?

   Hint: *Consider the two types of sushi separately and create an equation for each.*

9. In the diagram, the perimeter of the rectangle is 56 units. What is its area? *(Pascal 2008, Grade 9, #12)*



10. * Solve for x in the following equations:

    (a) $x^2 = 25$

    (b) $x^3 = 8$

    (c) $(x+1)(x-7) = 0$

11. * $50 is to be split among 3 friends. Friend B gets double the amount of friend A. Friend C gets $10 more than friend B. How much money does each friend get?

12. * In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum. In the magic square shown, what is the sum $a+b+c$? *(Pascal 2015, Grade 9, #18)*

a	13	b
19	c	11
12	d	16

13. * The average of a list of three consecutive odd integers is 7. When a fourth positive integer, m, different from the first three, is included in the list, the average of the list is an integer. What is the sum of the three smallest possible values of m? *(Cayley 2015, Grade 10, #21)*
Hint: *To calculate the* ***average*** *of a list of numbers: add up all the numbers, then divide by how many numbers there are.*

14. * Robert Wadlow, the tallest person in history lived from Feb. 22 1918 until July 15 1940. He was recorded as being 2.72 metres and 439 lbs at his time of death when he was 22 years old.

   (a) Assuming that he grew in height at a constant rate each year, and started as 0.00 m long (*Note:* This is most likely an unreasonable assumption but for calculation purposes we will continue with this assumption), how much did Robert grow each year (Round your answer to four decimal places)?

   (b) A more reasonable assumption would be that Robert was born with a height of 50 cm, grew 20 cm in the first year, 15 cm each year for the next 11 years and from there a constant growth rate until the age of 22. How much did he grow each year after he was 12 years old?