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Grade 6 Math Circles

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Exponentiation

Multiplication is often thought as *repeated addition.*

Exercise: Evaluate the following using multiplication.

$1+1+1+1+1 =$ __________ $\quad 7+7+7+7+7+7 =$ __________ $\quad 9+9+9+9 =$ __________

Exponentiation is *repeated multiplication.* Exponentiation is useful in areas like:

Computers, Population Increase, Business, Sales, Medicine, Earthquakes

An exponent and base looks like the following:

$$4^3$$

The small number written above and to the right of the number is called the ______________.

The number underneath the exponent is called the ______________.

In the example, 4^3, the exponent is ______ and the base is ______. 4 is raised to the **third power**.

An exponent tells us to multiply the base by itself that number of times.

In the example above, $4^3 = 4 \times 4 \times 4$. Once we write out the multiplication problem, we can easily evaluate the expression. Let's do this for the example we've been working with:

$$\begin{aligned} 4^3 &= 4 \times 4 \times 4 \\ 4^3 &= 16 \times 4 \\ 4^3 &= 64 \end{aligned}$$

The main reason we use exponents is because it's a shorter way to write out big numbers. For example, we can express the following long expression: $2 \times 2 \times 2 \times 2 \times 2 \times 2$ as 2^6 since 2 is being multiplied by itself 6 times. We say 2 is raised to the **6th power**.

Exercise: Express each of the following using an exponent.

$7 \times 7 \times 7 =$ ______ $5 \times 5 \times 5 \times 5 \times 5 \times 5 =$ ______ $10 \times 10 \times 10 \times 10 =$ ______

$9 \times 9 =$ ______ $1 \times 1 \times 1 \times 1 \times 1 =$ ______ $8 \times 8 \times 8 =$ ______

Exercise: Evaluate.

$2^5 =$ ______ $8^2 =$ ______ $6^3 =$ ______ $1^9 =$ ______

You may already have questions such as, "if we can exponentiate with any number, what about zero?" or, "what about one?" or even, "what about negative numbers?" Lets think about the answers to those questions.

Exercise: To answer some of the questions above, attempt the following:

Base of 1 or 0:

$1^2 =$ ______ $1^3 =$ ______ $1^{27} =$ ______ $1^{2019} =$ ______

$0^2 =$ ______ $0^3 =$ ______ $0^{27} =$ ______ $0^{2019} =$ ______

Exponent of 1 or 0:

$4^1 =$ ______ $35^1 =$ ______ $278^1 =$ ______ $2019^1 =$ ______

$4^0 =$ ______ $35^0 =$ ______ $278^0 =$ ______ $2019^0 =$ ______

Negative Base:

$(-1)^2 =$ ______ $(-1)^{2019} =$ ______ $(-2)^2 =$ ______ $(-6)^2 =$ ______

$(-1)^5 =$ ______ $(-1)^{2020} =$ ______ $(-2)^3 =$ ______ $(-6)^3 =$ ______

Negative Exponent:

$1^{-1} =$ ______ $(-2)^{-2} =$ ______ $2019^{-1} =$ ______ $8^{-2} =$ ______

$(-1)^{-1} =$ ______ $2^{-2} =$ ______ $(-2019)^{-1} =$ ______ $(-5)^{-3} =$ ______

Base of 0 and Exponent of 0:

$0^0 =$ ______

Special Cases

Case	Explanation	Formula	Examples
Base of 1	A base 1 with any exponent is 1.	$1^a = 1$	$1^{2468} = 1$
Base of 0	A base 0 with any exponent is 0.	$0^a = 0$	$0^{357} = 0$
Exponent of 1	Any base with exponent 1 is equal to base. Let **b** be any number.	$b^1 = b$	$8^1 = 8$ $(-2)^1 = -2$
Exponent of 0	Any base with exponent 0 is equal to 1. Let **b** be any number.	$b^0 = 1$	$13^0 = 1$ $(-9)^0 = 1$
Negative Base	Let **(-b)** be a negative base. Let **a** be an even exponent. Let **c** be an odd exponent. **Note**: $(+)(+) = (+)$ $(-)(-) = (+)$ $(-)(+) = (-)$ $(+)(-) = (-)$	$(-b)^a = b^a$ $(-b)^c = -(b^c)$	$(-1)^2 = 1$ $(-1)^3 = -1$ $(-2)^2 = 2^2 = 4$ $(-2)^3 = -(2^3) = -8$
Negative Exponent	Let **b** be any number (other than 0). Let **(-a)** be any negative exponent.	$b^{-a} = \frac{1}{b^a}$ $(-b)^{-a} = \frac{1}{(-b)^a}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ $(-3)^{-3} = \frac{1}{(-3)^3} = \frac{-1}{27}$ $(-4)^{-2} = \frac{1}{(-4)^2} = \frac{1}{16}$

Note that there is one case that is not talked about in the table above and that is 0^0.

From the table above, base 0 with any exponent is 0 which tells us that 0^0 should be equal to 0?

But from the table above, any base with exponent 0 is equal to 1 so then 0^0 should be equal to 1?

Base of 0 and Exponent of 0 (0^0)

The statement $0^0 = 1$ is **ambiguous** and has been long debated in mathematics.

Many sources consider 0^0 to be an **indeterminate form**, or say that 0^0 is **undefined**. On the other hand, other sources/branches of mathematics define $0^0 = 1$.

Note that, certainly, $0^0 \neq 0$. You can do some research to find out why!

Exponent Rules

Product Rule - Given a base **b** and two numbers **n**, **m**:

$$b^n \times b^m = b^{n+m}$$

Example: $2^2 \times 2^3 = 2^{2+3} = 2^5$

Quotient Rule - Given a base **b** and two numbers **n**, **m**:

$$\frac{b^n}{b^m} = b^{n-m}$$

Example: $\frac{3^9}{3^6} = 3^{9-6} = 3^3$

Power Rule - Given a base **b** and two numbers **n**, **m**:

$$(b^n)^m = (b^m)^n = b^{m\times n}$$

Example: $(2^2)^3 = (2^3)^2 = 2^{3\times 2} = 2^6$

Power of a Product - Given two bases **a** and **b** and an exponent **m**:

$$(ab)^m = (a \times b)^m = a^m \times b^m$$

Example: $(3 \times 4)^2 = 3^2 \times 4^2$

Power of a Quotient - Given two bases **a** and **b** and an exponent **m**:

$$(\frac{a}{b})^m = \frac{a^m}{b^m}$$

Example: $(\frac{2}{3})^2 = \frac{2^2}{3^2}$

Exercise Set

"*" indicates challenge questions.

1. Simplify each product.

   (a) $10^{12} \times 10^{35}$ (b) $a^7 \times a^{12}$ (c) $c^3 \times c^8$

   (d) $w^{103} \times w^{103}$ (e) $10^a \times 10^b$ (f) $d^7 \times d^{-7}$

   (g) $(3m^5)(7m^{19})$ (h)* $g^{12} \times g^{19} \times g^{11}$ (i)* $(2a^3b^4)(5a^6b^7)$

2. Simplify each expression.

   (a) $(2^3)^2$ (b) $(5^4)^2$ (c) $(x^2)^3$

   (d) $[(-1)^5]^{20}$ (e) $(2^{-4})^5$ (f) $(4y^3)^2$

   (g) $(-3h^9)^3$ (h) $(y \times d^6)^8$ (i)* $(4h^3)^2(-2g^3h)^3$

3. Simplify each quotient and evaluate.

   (a) $\dfrac{10^6}{10^2}$ (b) $\dfrac{4^{17}}{4^{14}}$ (c) $\dfrac{9^{210}}{9^{207}}$ (d)* $\dfrac{8^{r+4}}{8^{r+1}}$

4. Simplify each quotient and evaluate if possible.

   (a) $(\dfrac{3}{7})^2$ (b) $(\dfrac{x}{y})^6$ (c)* $(\dfrac{4d^3}{c^5})^3$ (d)* $(\dfrac{8m^{11}n^4}{m^3})^2$

5. Complete the inequality with >, <, or =.

   (a) 2^5 ______ 5^2

   (b) 6^0 ______ 78^0

   (c) 10^3 ______ 3^{10}

   (d) 4^8 ______ 4^{-8}

   (e) 2^4 ______ 4^2

   (f) * 2^{-2} ______ 2^{-4}

   (g) * 3^{-19} ______ 3^{-5}

Logarithms

Logarithms are another way of thinking about exponents.

For example, we know that 2 raised to the 4^{th} power equals 16. This is expressed by the **exponential** equation $2^4 = 16$.

Now suppose someone asked us, "2 raised to which power equals 16?" The answer would be 4. This is expressed by the **logarithmic** equation $\log_2(16) = 4$, read as "*log base two of sixteen is four*".

$$2^4 = 16 \iff \log_2(16) = 4$$

Both equations describe the same relationship between 2, 4, and 16, where 2 is the **base** and 4 is the **exponent**. More examples are listed in the table below.

Logarithmic Form		Exponential Form
$\log_2(8) = 3$	$\iff$	$2^3 = 8$
$\log_3(81) = 4$	$\iff$	$3^4 = 81$
$\log_5(25) = 2$	$\iff$	$5^2 = 25$

Logarithms can be defined as:

$$\log_b(a) = c \iff b^c = a$$

Both equations describe the same relationship between a, b, and c:

- b is the base,
- c is the exponent, and
- a is called the argument.

It is helpful to remember that the base of the logarithm is the same as the base of the exponent.

Exercise: Write the equivalent logarithmic equation of the following:

$2^5 = 32 \iff$ ____________________ $5^3 = 125 \iff$ ____________________

Exercise: Write the equivalent exponential equation of the following:

$\log_2(64) = 6 \iff$ ____________________ $\log_4(16) = 2 \iff$ ____________________

Evaluating Logarithms

Now that we understand the relationship between exponents and logarithms, we can try to evaluate some simple logarithms.

For example, let's evaluate $\log_4(64)$.

Let's start by setting that expression equal to x to get:

$$\log_4(64) = x$$

Writing this in exponential form gives us:

$$4^x = 64$$

4 to what power is 64?

We can easily check that $4^3 = 64$ so $\log_4(64) = 3$.

Exercise: Evaluate each of the following.

Remember, when evaluating $\log_b(a)$, you can ask: "b to what power is a?

1. $\log_6(36) =$ ____________
2. $\log_3(27) =$ ____________
3. $\log_4(4) =$ ____________
4. * $\log_5(1) =$ ____________
5. * $\log_2(\frac{1}{2}) =$ ____________
6. * $\log_3(\frac{1}{9}) =$ ____________

Why study logarithms?

As you just learned, logarithms reverse exponents. For this reason, they are very helpful for solving exponential equations.

Logarithmic expressions and functions also turn out to be very interesting by themselves, and are actually very common in the world around us. For example, many physical phenomena are measured with logarithmic scales.

Later on, you can learn about the properties of logarithms that help us rewrite logarithmic expressions, and about the **change of base rule** that allows us to evaluate any logarithm we want using the calculator.

Problem Set:

1. Evaluate.

   (a) 8^2
   (b) 4^{-1}
   (c) $7^{-6} \times 7^8$
   (d) 3^3
   (e) 27^0
   (f) * 2^{3^2}
   (g) 1^{337}
   (h) $18 - 2^3$
   (i) * $\dfrac{2(-2)^4 \times 3^4 \times 2}{3 \times 2^5}$

2. Write the following as exponents.

   (a) $4 \times 4 \times 4$
   (b) 7 to the fifth power
   (c) $(-9) \times 4 \times 4 \times (-9) \times 4$
   (d) $3 \times 3 \times 3 \times 3 \times 3 \times 3$
   (e) 89 to the second power
   (f) * $\dfrac{-1}{216}$

3. Write the equivalent logarithmic equation of each exponential equation. Write the equivalent exponential equation of each logarithmic equation.

   (a) $\log_7(7) = 1$
   (b) $3^{-4} = \dfrac{1}{81}$
   (c) $(\dfrac{1}{2})^5 = \dfrac{1}{32}$
   (d) * $\log_{\frac{1}{2}}(\dfrac{1}{8}) = 3$
   (e) * $x^{2z} = y$
   (f) * $\log_m(n) = 2$

4. The Sun is about $5^2 \times 5^2$ million miles away from Earth. Write $5^2 \times 5^2$ using an exponent. How many miles away is the Sun?

5. An asteroid travels at a speed of 6^8 miles per day, how many miles will it travel in 6^3 day?

6. The population of giraffes triples every year, how many giraffes will there be in 5 years if there are currently 2?

7. The population of a bacteria decreases by half every 30 minutes. If there are initially 128 bacteria, how many bacteria will be there after 1.5 hours?

8. A **polynomial** is an expression involving x and its various powers. Evaluate each polynomial for the given value of x.

   (a) $x^2 - 64$, where $x = -8$

   (b) $x^2 - 6x + 8$ where , $x = 2$

   (c) $8x^7 + 7x^6 + 6x^5 + 5x^4 + 4x^3 + 3x^2 + 2x$, where $x = 0$

   (d) $250x^{12} + 340x^{10} + 670x^{13}$, where $x = 1$

9. * Solve for x.

   Hint: *Write the greater base in terms of the smaller base and use Product Rule.*

   (a) $4^x = 64^2$

   (b) $3^6 = 27^x$

   (c) $5^{33} = 125^x$

   (d) * $6^{x+2} = 216$

   (e) * $8^{x-1} = 2^6$

   (f) * $10^x - 10 = 9990$

10. * Express $\dfrac{16^4 \times 64^3}{2^{24}}$ as a power of base 2.

11. ** If you have $1 \leq 10^n \leq 1,000,000,000$, what is the maximum value of 3^{-n}?

    Hint: *The ≤ symbol means less than or equal to. This means, 3 ≤ 4 since it is less than 4 but also 4 ≤ 4 since it is equal to 4.*