UNIVERSITY OF WATERLOO

Faculty of Mathematics

Waterloo, Ontario

Centre for Education in

Mathematics and Computing

# Grade 6 Math Circles

Fall $\{20, 19\}$ - Oct 29/30

## *Sequences*

**Sequences** are ordered lists of numbers. Sequences are useful in areas like:

**Population Growth, Probability, Statistics, Physics (Bouncing Ball), Nature**

Each number in the list is called a **term**. All terms are related by a rule or a **pattern** that allows us to predict the next term. All terms in sequences may be expressed inside curly brackets.

**Example**: {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, ... }.

We can describe this sequence in multiple ways:

1. Increasing odd positive integers.

2. Each new term is two more than the previous term.

### Notation

Each number in the sequence will be expressed by $a_n$ where $n$ is the term number in the sequence.

**Example**: {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, ... }.

$a_1 = 1$ or in other words, the 1st term in the sequence is 1.

$a_7 = 13$ or in other words, the 7th term in the sequence is 13.

**Exercise**: Identify the pattern and extend the sequences according to their pattern.

1. 3, 11, 19, 27, ______
2. 38, 35, 32, 29, ______
3. 1, 2, 4, 8, ______
4. 81, 27, 9, 3, ______
5. 5, 5, 5, 5, 5, ______
6. $0, \frac{-1}{2}, -1, \frac{-3}{2}, -2,$ ______

## Recursive Sequences

Consider the following sequence:

$$\{3, 2, 5, 7, 12, 19, ...\}$$

Given $a_1$ and $a_2$, every term after is produced by adding the previous two terms. The third term, $a_3 = 5$ is the sum of $a_1$ and $a_2$, $a_4 = 7$ is the sum of $a_2$ and $a_3$ and so on. Using this, $a_n = a_{n-1} + a_{n-2}$. This is an example of a *recursive sequence*.

A **recursive sequence** is a sequence in which terms are defined using one or more previous terms which are given.

**Exercises**

*"*" indicates a challenge question*

1. Consider the sequence {-4, -3, -7, -10, -17, -27, ...}. What is the next term in the sequence?

2. For a sequence, $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$ and the pattern is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. What are the next 3 terms in the sequence?

3. * Consider the sequence {1, 4, 8, 13, 19, 26, ...}. What is the next term in the sequence?
   ***Hint***: *The rule / pattern here is not consistent.*

4. ** Consider the sequence {1, 2, 2, 4, 8, 11, 33, 37, 148, ...}. What is the next term?
   ***Hint***: *The rule / pattern here is not consistent.*

## Arithmetic Sequences

**Arithmetic Sequences** are recursive sequences with patterns that involve adding or subtracting a *constant* value to each term to get the next term.

In an arithmetic sequence, the difference between consecutive terms is always equal.

**Example**:

1. $\{3, \xrightarrow{+2} 5, \xrightarrow{+2} 7, \xrightarrow{+2} 9, ...\}$ add a constant value of 2 to each term to get the next
2. $\{21, \xrightarrow{-5} 16, \xrightarrow{-5} 11, \xrightarrow{-5} 6, ...\}$ subtract a constant value of 5 from each term to get the next

**Common Difference**

The **common difference** of an arithmetic sequence is the difference between two consecutive terms.

**Example**:

1. The common difference of {10, 21, 32, 43, ...} is 11.
2. The common difference of {-2, -5, -8, -11, ...} is -3.

**Exercises**:

1. Select all arithmetic sequences and determine the common difference of each.

   (a) {13, 16, 19, 22, ...}

   (b) {2, 4, 8, 16, 32, ...}

   (c) {0, 1, 3, 6, 10, ...}

   (d) {37, 33, 29, 25, ...}

   (e) {5, 2, -1, -4, ...}

   (f) $\{-2, \frac{-5}{3}, \frac{-4}{3}, -1, \frac{-2}{3}, ...\}$

2. The first term of an arithmetic sequence is 10 and its common difference is negative seven. What is the fifth term of the sequence?

3. The fourth term of an arithmetic sequence is 27 and its common difference is negative five. What is the first term of the sequence?

## Arithmetic Sequences Formula

**Formulas** give us instructions on how to find any term of a sequence.

To remain general, formulas use $n$ to represent the term number and $a_n$ to represent the n$^{th}$ term of the sequence. **Formulas** tell us how to find $a_n$ for any possible n.

**Example**: For the arithmetic sequence {5, 9, 13, 17, 21, 25, 29, ...}, the 3$^{rd}$ term is 13 or in other words, $a_3 = 13$.

In the sequence above, we are able to find a pattern between the terms of the sequence and the first term as follows:

$$\{5,\ 5+4,\ 5+8,\ 5+12,\ 5+16,\ 5+20,\ 5+24,\ ...\}$$

Now let's write each expression using the common difference, 4.

$$\{5,\ 5+4(1),\ 5+4(2),\ 5+4(3),\ 5+4(4),\ 5+4(5),\ 5+4(6),\ ...\}$$

Are we able to generalize each term using the common difference 4? Let's define $a_n = 5 + 4(n - 1)$.

$$a_1 = 5 + 4(0) = 5 \quad 1^{st} \text{ term}$$

$$a_2 = 5 + 4(1) = 9 \quad 2^{nd} \text{ term}$$

$$...$$

$$a_7 = 5 + 4(6) = 29 \quad 7^{th} \text{ term}$$

**Arithmetic Sequences Formula**

The formula for the n$^{th}$ term of an arithemtic sequence is as follows:

$$a_n = a_1 + d(n - 1) \text{ , where}$$

- $a_1$ is the first term in the sequence
- $d$ is the common difference of the sequence

**Exercise**: Find the 4$^{th}$ term in the sequence defined by $a_n = -6 - 4(n - 1)$.

**Exercise**: Determine the formula for the arithmetic sequence $\{\frac{4}{5}, \frac{7}{5}, 2, \frac{13}{5}, ...\}$.

## Geometric Sequences

**Geometric Sequences** are recursive sequences with patterns that involve multiplying each term by a *non-zero constant* value to get the next term. The constant value is the **common ratio**.

**Example**:

1. $\{2, \xrightarrow{\times 3} 6, \xrightarrow{\times 3} 18, \xrightarrow{\times 3} 54, ...\}$ with common ratio +3.

2. $\{7, \xrightarrow{\times(-1)} \text{-}7, \xrightarrow{\times(-1)} 7, \xrightarrow{\times(-1)} \text{-}7, ...\}$ with common ratio (-1).

Let's consider the following geometric sequence.

$$\{64, 32, 16, 8, 4, 2, ...\}$$

Notice that here we are **dividing** each term by a constant value of 2. However, we can write this in terms of a multiplication simply by treating the division as a fraction multiplication. Here we are dividing each term by 2 but if we multiplied each term by $\frac{1}{2}$, we would get the same results. So the common ratio is $\frac{1}{2}$.

To **determine the common ratio**, $c$, we simply divide $a_n$ by $a_{n-1}$ for any $n$ to get $c = \frac{a_n}{a_{n-1}}$.

**Example**: In the geometric sequence {2, 6, 18, 54, ...} given above, the common ratio is 3. We can verify using the method above as follows:

$$\frac{54}{18} = \frac{18}{6} = \frac{6}{2} = +3$$

**Exercise**: Determine the common ratio of the geometric sequence {3, 6, 12, 24, 48, 96, 192, ...}.

**Exercise**: Determine the common ratio of the geometric sequence {1296, -216, 36, -6, ...}. What is the next term in the sequence?

## Geometric Sequences Formula

Going back to the idea of formulas, we use $n$ to represent the term number and $a_n$ to represent the n$^{th}$ term of the sequence.

**Example**: For the geometric sequence {2, 4, 8, 16, 32, 64, 128...}, the 4$^{th}$ term is 16 or in other words, $a_4 = 16$.

In the sequence above, we are able to find a pattern between each two consecutive terms:

$$\{2, 2 \times 2, 4 \times 2, 8 \times 2, 16 \times 2, 32 \times 2, 64 \times 2, ...\}$$

Are we able to generalize each term using its previous term? Let's define:

$$\begin{cases} a_1 = 2 & \text{the first term} \\ a_n = a_{n-1} \times 2 & \text{all remaining terms} \end{cases}$$

We use this formula to check the sequence above:

$$\begin{aligned} a_1 &= 2 && 1^{st} \text{ term} \\ a_2 &= a_1 \times 2 \\ &= 2 \times 2 = 4 && 2^{nd} \text{ term} \\ a_3 &= a_2 \times 2 \\ &= 4 \times 2 = 8 && 3^{rd} \text{ term} \\ & ... \end{aligned}$$

**Geometric Sequences Formula**

Given $a_1$ as the first term, the formula for the n$^{th}$ term of a geometric sequence is as follows:

$$a_n = a_{n-1} \times c \text{ , where}$$

- $a_{n-1}$ is the previous term in the sequence
- $c$ is the common ratio of the sequence

*You will notice it is more tedious to find the n$^{th}$ term using this formula than the Arithmetic Sequence formula. There are ways to make this more efficient which you will learn in future years!*

## Exercise Set

"*" indicates challenge questions.

1. Extend the following geometric sequences.

   (a) 3, 6, 12, ______

   (b) 375, 75, 15, ______

   (c) 7, 21, 63, ______

   (d) 24, 12, 6, ______

   (e) 5, 20, 80, ______

   (f) $\frac{-1}{32}, \frac{1}{16}, \frac{-1}{8}$, ______

2. Determine the formula for the geometric sequence $\{64, 8, 1, \frac{1}{8}, ...\}$.

3. Determine the formula for the arithmetic sequence $\{\frac{4}{5}, \frac{7}{5}, 2, \frac{13}{5}, ...\}$.

4. A geometric sequence is defined by $a_1 = \frac{-1}{8}$ and $a_n = 2 \times a_{n-1}$. What is $a_4$, the $4^{th}$ term in the sequence?

5. A geometric sequence is defined by $a_1 = 15$ and $a_n = -3 \times a_{n-1}$. What is $a_4$, the $4^{th}$ term in the sequence?

6. We can define an arithmetic sequence as follows:

$$\begin{cases} a_1 = 2 & \text{the first term} \\ a_n = a_{n-1} - 6 & \text{all remaining terms} \end{cases}$$

   Determine the first 5 terms in the sequence.

## Special Sequences

A **special sequence** is a sequence that has a unique pattern to it.

### Sequence of Square Numbers

A **square number** is a number that results from multiplying a number by itself.

$$1 \times 1 = 1 \qquad 2 \times 2 = 4 \qquad 3 \times 3 = 9 \qquad etc.$$

The sequence of square numbers can be written as follows:

$$\{1, 4, 9, 16, 25, 36, 49, ...\}$$

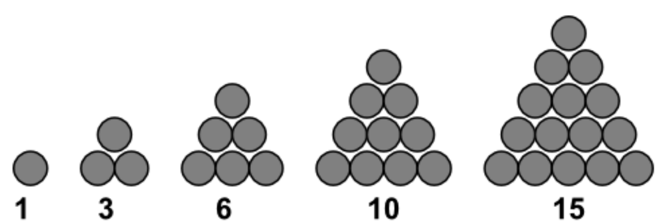
We can observe the following pattern between each term and the term number

$$\begin{aligned} 1 \times 1 &= 1 \quad 1^{st} \text{ term} \\ 2 \times 2 &= 4 \quad 2^{nd} \text{ term} \\ 3 \times 3 &= 9 \quad 3^{rd} \text{ term} \\ 4 \times 4 &= 16 \quad 4^{th} \text{ term} \end{aligned}$$

We can continue the pattern above to conclude that for the $n^{th}$ term, $a_n = n \times n$.

### Triangular Sequence

A **triangular sequence** is a sequence that gives the number needed to form a triangle. Observe the diagram below:




The number of circles needed to make each triangle in order can be written as:

$$\{1, 3, 6, 10, 15, ...\}$$

For the pattern above, we can conclude that the $n^{th}$ term can be written as $a_n = a_{n-1} + n$ with $a_1 = 1$.

**Exercise**: Following the pattern, how many circles would you need to make the next triangle?

# Special Sequences Continued

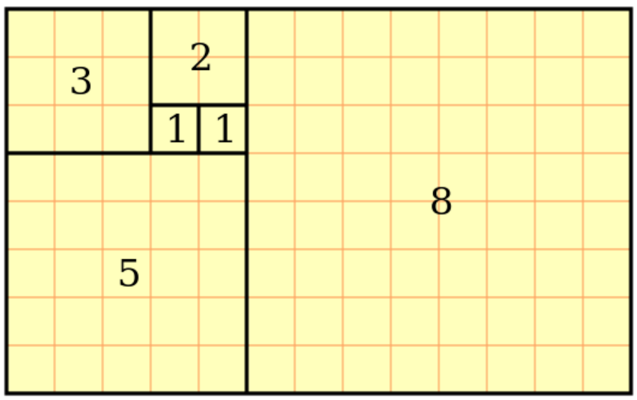
## Fibonacci Sequence

The **Fibonacci sequence** is a recursive sequence where each term is generated by adding the two previous numbers in the sequence:

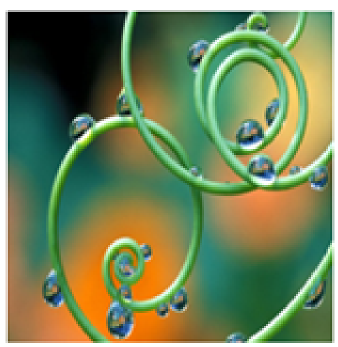
$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...\}$$

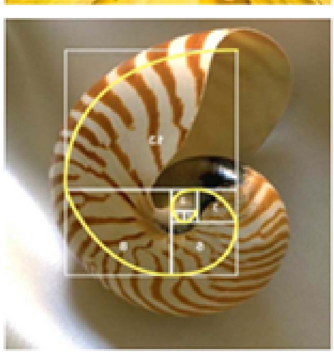
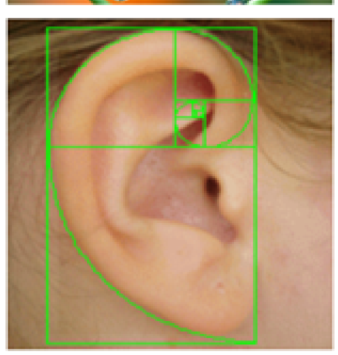
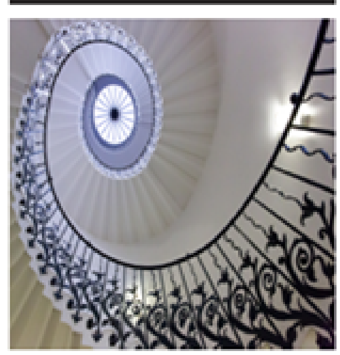
The formula for the Fibonacci sequence can be written as $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 0$ and $a_2 = 1$. The Fibonacci sequence appears in the world around us. Let's see how!

**Exercise**: Complete the **Fibonacci spiral** in the grid below.




We can now observe how this spiral appears all around us.

## Problem Set:

1. Complete the recursive formula for each sequence.

   (a) {12, 10, 8, 6, ...}

   $$\begin{cases} a_1 = \_\_\_\_\_\_\_\_\_\_ \\ a_n = \_\_\_\_\_\_\_\_\_\_ \end{cases}$$

   (b) {-15, -90, -540, ...}

   $$\begin{cases} h_1 = \_\_\_\_\_\_\_\_\_\_ \\ h_n = \_\_\_\_\_\_\_\_\_\_ \end{cases}$$

   (c) {300, 60, 12, 2.4, ...}

   $$\begin{cases} d_1 = \_\_\_\_\_\_\_\_\_\_ \\ d_n = \_\_\_\_\_\_\_\_\_\_ \end{cases}$$

2. Write the formula for $a_n$ for each of the following sequences.

   ***Hint***: *For geometric sequences, remember to include the first term.*

   (a) {-7, -2, 3, 8, 13, 18, ...}

   (b) {200, 100, 50, 25, ... }

   (c) {170, 85, 0, -85, ...}

   (d) {96, 24, 6, 1.5, ...}

   (e) * $\{\frac{5}{32}, 1, \frac{59}{32}, \frac{86}{32}, ...\}$

   (f) * $\{6, 4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, ...\}$

3. Determine the indicated term for each sequence.

   (a) $a_1 = \frac{3}{16}$ and $a_n = a_{n-1} \times 4$ $\quad a_3 =$ __________

   (b) $b_1 = 0$, $b_2 = 5$, and $b_n = b_{n-1} + b_{n-2}$ $\quad b_4 =$ __________

   (c) $c_1 = 3$ and $c_n = c_{n-1} - 14$ $\quad c_3 =$ __________

   (d) $d_1 = \frac{2}{7}$ and $d_n = d_{n-1} \times 7$ $\quad d_5 =$ __________

   (e) $e_1 = 0$ and $e_n = e_{n-1} \times \frac{13}{49}$ $\quad e_{17} =$ __________

   (f) $f_1 = 18$ and $f_n = f_{n-1} \times \frac{1}{6}$ $\quad f_6 =$ __________

4. Determine the missing terms in each sequence.

   (a) -8, -14, -20, ______, ______

   (b) 189, 63, 21, ______, ______

   (c) 15, 14, 29, ______, 72, 115, 187

   (d) ______, 64, 49, 36, ______, 16, 9, ______, ______

   (e) $\frac{1}{96}, \frac{1}{16},$ ______, $1, 4$

   (f) * 0, 2, 5, ______, 14, 20, 27

   (g) * ______, $\frac{27}{16}, \frac{-9}{4}, 3,$ ______

   (h) * 415, 257, 158, 99, 59, 40, ______

5. Find the common difference and the $n^{th}$ term.

   (a) {89, 78, 67, 56, 45, ...}

   (b) {55, 62, 69, 76, 83, ...}

   (c) {27, 27, 27, 27, 27, ...}

   (d) {0.3, 0.5, 0.7, 0.9, 1.1, ...}

6. Find the common ratio and the next term.

   (a) {15, 45, 135, 405 ...}

   (b) {4096, 1024, 256, 64 ...}

   (c) {1, 1, 1, 1, 1, ...}

   (d) * {1, $\sqrt{5}$, 5, ...}

7. Scott has decided to add strength training to his exercise program. His trainer suggests that he add weight lifting for 5 minutes during his routine for the first week. Each week thereafter, he is to increase the weight lifting time by 2 minutes. If Scott continues with this increase in weight lifting time, how many minutes will he be devoting to weight lifting in week 10?

8. A research lab is to begin experimentation with a bacteria that doubles every 4 hours. The lab starts with 200 bacteria. How many bacteria will be present at the end of the $12^{th}$ hour?

9. Your father wants you to help him build a shed in the backyard. He says he will pay you $10 for the first week and add an additional $10.50 each week thereafter. The project will take 5 weeks. How much money will you earn, in total, if you work for the 5 weeks?

10. The summer Olympics occur every four years. Starting with 2016, in which year will the $12^{th}$ summer Olympics occur?

11. The terms in the sequence {2, 7, 12, 17, 22, ..} increase by 5. The terms in the sequence {3,10, 17, 24, 31, ...} increase by 7. The number 17 occurs in **both** sequences. What is the next number that appears in **both** sequences?

12. In the sequence 32, 8, ______, ______, x, each term after the second is the average of the two terms immediately before it. What is the value of x? *(Pascal 2005, Grade 9. #10)*
    ***Hint***: *The average of two numbers is their sum divided by 2.*

13. Arya plays Candy Crush every day for 7 days. Each day after the first, he plays 5 more levels than before. In total, he played 175 levels. How many levels did he play on the last day?

14. * Penelope folds a piece of paper in half, creating two layers of paper. She folds the paper in half again, creating a total of four layers of paper. If she continues to fold the paper in half, which of the following is a possible number of layers that could be obtained?
    *(Cayley 2017, Grade 10, #6)*
    (a) 10 (b) 12 (c) 14 (d) 16 (e) 18

15. * What is the sum of the first 2005 terms of the sequence 1, 2, 3, 4, 1, 2, 3, 4, ...?
    *(Fermat 2005, Grade 11, #6)*

16. * Seven children, each with the same birthday, were born in seven consecutive years. The sum of the ages of the youngest three children is 42. What is the sum of the ages of the oldest three? *(Fermat 2005, Grade 11, #8)*

17. * When expressed as a decimal, $\frac{1}{7} = 0.142857142857...$ . Which of the following positions to the right of the decimal will be a 2? *(Gauss 2015, Grade 8, #17)*
    (a) $119^{th}$ (b) $121^{st}$ (c) $123^{rd}$ (d) $125^{th}$ (e) $126^{th}$

18. ** What is the tens digit of $3^{2016}$? *(Gauss 2016, Grade 8, #24)*