UNIVERSITY OF WATERLOO

Faculty of Mathematics

Waterloo, Ontario

Centre for Education in Mathematics and Computing

Grade 6 Math Circles

Fall $\{20, 19\}$ - Oct 29/30

Sequences Solutions

Sequences are ordered lists of numbers. Sequences are useful in areas like:

Population Growth, Probability, Statistics, Physics (Bouncing Ball), Nature

Each number in the list is called a **term**. All terms are related by a rule or a **pattern** that allows us to predict the next term. All terms in sequences may be expressed inside curly brackets.

Example: {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, ... }.

We can describe this sequence in multiple ways:

1. Increasing odd positive integers.
2. Each new term is two more than the previous term.

Notation

Each number in the sequence will be expressed by a_n where n is the term number in the sequence.

Example: {1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, ... }.

$a_1 = 1$ or in other words, the 1st term in the sequence is 1.

$a_7 = 13$ or in other words, the 7th term in the sequence is 13.

Exercise: Identify the pattern and extend the sequences according to their pattern.

1. 3, 11, 19, 27, 35 (+8)
2. 38, 35, 32, 29, 26 (-3)
3. 1, 2, 4, 8, 16 (×2)
4. 81, 27, 9, 3, 1 (÷3)
5. 5, 5, 5, 5, 5, 5 (+0)
6. $0, \frac{-1}{2}, -1, \frac{-3}{2}, -2, \frac{-5}{2}$ $(-\frac{1}{2})$

Recursive Sequences

Consider the following sequence:

$$\{3, 2, 5, 7, 12, 19, ...\}$$

Given a_1 and a_2, every term after is produced by adding the previous two terms. The third term, $a_3 = 5$ is the sum of a_1 and a_2, $a_4 = 7$ is the sum of a_2 and a_3 and so on. Using this, $a_n = a_{n-1} + a_{n-2}$. This is an example of a *recursive sequence*.

A **recursive sequence** is a sequence in which terms are defined using one or more previous terms which are given.

Exercises

"" indicates a challenge question*

1. Consider the sequence {-4, -3, -7, -10, -17, -27, ...}. What is the next term in the sequence?

   Observe that $a_n = a_{n-1} + a_{n-2}$. The next term in the sequence is a_7 and so:

   $$a_7 = a_6 + a_5 = -17 - 27 = -44$$

2. For a sequence, $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$ and the pattern is $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. What are the next 3 terms in the sequence?

   $a_4 = a_3 + a_2 + a_1 = 3 + 2 + 1 = 6$

   $a_5 = a_4 + a_3 + a_2 = 6 + 3 + 2 = 11$

   $a_6 = a_5 + a_4 + a_3 = 11 + 6 + 3 = 20$

   Then the sequence can be written as {1, 2, 3, 6, 11, 20, ...}.

3. * Consider the sequence {1, 4, 8, 13, 19, 26, ...}. What is the next term in the sequence?

   Hint: *The rule / pattern here is not consistent.*

   Observe that $1 \xrightarrow{+3} 4 \xrightarrow{+4} 8 \xrightarrow{+5} 13 \xrightarrow{+6} 19 \xrightarrow{+7} 26$.

   So the next term should be $26 + 8 = \mathbf{34}$.

4. ** Consider the sequence {1, 2, 2, 4, 8, 11, 33, 37, 148, ...}. What is the next term?

   Hint: *The rule / pattern here is not consistent.*

   Observe that $1 \xrightarrow{+1} 2 \xrightarrow{\times 1} 2 \xrightarrow{+2} 4 \xrightarrow{\times 2} 8 \xrightarrow{+3} 11 \xrightarrow{\times 3} 33 \xrightarrow{+4} 37 \xrightarrow{\times 4} 148$.

   So the next term should be $148 + 5 = \mathbf{153}$.

Arithmetic Sequences

Arithmetic Sequences are recursive sequences with patterns that involve adding or subtracting a *constant* value to each term to get the next term.

In an arithmetic sequence, the difference between consecutive terms is always equal.

Example:

1. $\{3, \xrightarrow{+2} 5, \xrightarrow{+2} 7, \xrightarrow{+2} 9, ...\}$ add a constant value of 2 to each term to get the next
2. $\{21, \xrightarrow{-5} 16, \xrightarrow{-5} 11, \xrightarrow{-5} 6, ...\}$ subtract a constant value of 5 from each term to get the next

Common Difference

The **common difference** of an arithmetic sequence is the difference between two consecutive terms.

Example:

1. The common difference of {10, 21, 32, 43, ...} is 11.
2. The common difference of {-2, -5, -8, -11, ...} is -3.

Exercises:

1. Select all arithmetic sequences and determine the common difference of each sequence.

   (a) {13, 16, 19, 22, ...}, $d = +3$

   (b) {2, 4, 8, 16, 32, ...}

   (c) {0, 1, 3, 6, 10, ...}

   (d) {37, 33, 29, 25, ...}, $d = -4$

   (e) {5, 2, -1, -4, ...}, $d = -3$

   (f) $\{-2, \frac{-5}{3}, \frac{-4}{3}, -1, ...\}$, $d = +\frac{1}{3}$

2. The first term of an arithmetic sequence is 10 and its common difference is negative seven. What is the fifth term of the sequence?

   $\{10, \xrightarrow{-7} 3, \xrightarrow{-7} -4, \xrightarrow{-7} -11, \xrightarrow{-7} -18, ...\}$. The 5^{th} term of the sequence is **18**.

3. The fourth term of an arithmetic sequence is 27 and its common difference is negative five. What is the first term of the sequence?

   The sequence is $\{42, \xrightarrow{-5} 37, \xrightarrow{-5} 32, \xrightarrow{-5} 27, ...\}$. The 1^{st} term of the sequence is **42**.

Arithmetic Sequences Formula

Formulas give us instructions on how to find any term of a sequence.

To remain general, formulas use n to represent the term number and a_n to represent the nth term of the sequence. **Formulas** tell us how to find a_n for any possible n.

Example: For the arithmetic sequence {5, 9, 13, 17, 21, 25, 29, ...}, the 3rd term is 13 or in other words, $a_3 = 13$.

In the sequence above, we are able to find a pattern between the terms of the sequence and the first term as follows:

$$\{5,\ 5+4,\ 5+8,\ 5+12,\ 5+16,\ 5+20,\ 5+24,\ ...\}$$

Now let's write each expression using the common difference, 4.

$$\{5,\ 5+4(1),\ 5+4(2),\ 5+4(3),\ 5+4(4),\ 5+4(5),\ 5+4(6),\ ...\}$$

Are we able to generalize each term using the common difference 4? Let's define $a_n = 5 + 4(n - 1)$.

$$a_1 = 5 + 4(0) = 5 \quad 1^{st} \text{ term}$$

$$a_2 = 5 + 4(1) = 9 \quad 2^{nd} \text{ term}$$

$$\ldots$$

$$a_7 = 5 + 4(6) = 29 \quad 7^{th} \text{ term}$$

Arithmetic Sequences Formula

The formula for the nth term of an arithemtic sequence is as follows:

$$a_n = a_1 + d(n - 1) \text{ , where}$$

- a_1 is the first term in the sequence
- d is the common difference of the sequence

Exercise: Find the 4th term in the sequence defined by $a_n = -6 - 4(n - 1)$.

$a_4 = -6 - 4(4 - 1) = -6 - 4(3) = -6 - 12 =$ **-18**

Exercise: Determine the formula for the arithmetic sequence $\{\frac{4}{5}, \frac{7}{5}, 2, \frac{13}{5}, ...\}$.

$a_1 = \frac{4}{5}$ and each term is increasing by $+\frac{3}{5}$ so the formula is $a_n = \frac{4}{5} + \frac{3}{5}(n - 1)$.

Geometric Sequences

Geometric Sequences are recursive sequences with patterns that involve multiplying each term by a *non-zero constant* value to get the next term. The constant value is the **common ratio**.

Example:

1. $\{2, \xrightarrow{\times 3} 6, \xrightarrow{\times 3} 18, \xrightarrow{\times 3} 54, ...\}$ with common ratio +3.
2. $\{7, \xrightarrow{\times(-1)} \text{-}7, \xrightarrow{\times(-1)} 7, \xrightarrow{\times(-1)} \text{-}7, ...\}$ with common ratio (-1).

Let's consider the following geometric sequence.

$$\{64, 32, 16, 8, 4, 2, ...\}$$

Notice that here we are **dividing** each term by a constant value of 2. However, we can write this in terms of a multiplication simply by treating the division as a fraction multiplication. Here we are dividing each term by 2 but if multiplied each term by $\frac{1}{2}$, we would get the same results. So the common ratio is $\frac{1}{2}$.

To **determine the common ratio**, c, we simply divide a_n by a_{n-1} for any n to get $c = \frac{a_n}{a_{n-1}}$.

Example: In the geometric sequence {2, 6, 18, 54, ...} given above, the common ratio is 3. We can verify using the method above as follows:

$$\frac{54}{18} = \frac{18}{6} = \frac{6}{2} = +3$$

Exercise: Determine the common ratio of the geometric sequence {3, 6, 12, 24, 48, 96, 192, ...}.
Using $c = \frac{a_n}{a_{n-1}}$, the common ratio of this sequence is $c = \frac{24}{12} = \frac{6}{3} = 2$. Each term is multiplied by 2 to get the next term in the sequence.

Exercise: Determine the common ratio of the geometric sequence {1296, -216, 36, -6, ...}. What is the next term in the sequence?
Using $c = \frac{a_n}{a_{n-1}}$, the common ratio of this sequence is $c = \frac{-6}{36} = \frac{-1}{6}$.
The next term in the sequence is a_5. Using the formula, $a_5 = a_4 \times \frac{-1}{6} = -6 \times \frac{-1}{6} = \mathbf{+1}$.

Geometric Sequences Formula

Going back to the idea of formulas, we use n to represent the term number and a_n to represent the nth term of the sequence.

Example: For the geometric sequence {2, 4, 8, 16, 32, 64, 128...}, the 4^{th} term is 16 or in other words, $a_4 = 16$.

In the sequence above, we are able to find a pattern between each two consecutive terms:

$$\{2, 2\times2, 4\times2, 8\times2, 16\times2, 32\times2, 64\times2, ...\}$$

Are we able to generalize each term using its previous term? Let's define:

$$\begin{cases} a_1 = 2 & \text{the first term} \\ a_n = a_{n-1}\times 2 & \text{all remaining terms} \end{cases}$$

We use this formula to check the sequence above:

$$\begin{aligned} a_1 &= 2 && 1^{st}\text{ term} \\ a_2 &= a_1 \times 2 \\ &= 2\times2 = 4 && 2^{nd}\text{ term} \\ a_3 &= a_2\times 2 \\ &= 4\times 2 = 8 && 3^{rd}\text{ term} \\ &\ldots \end{aligned}$$

Geometric Sequences Formula

Given a_1 as the first term, the formula for the nth term of a geometric sequence is as follows:

$$a_n = a_{n-1}\times c\text{ , where}$$

- a_{n-1} is the previous term in the sequence
- c is the common ratio of the sequence

You will notice it is more tedious to find the nth term using this formula than the Arithmetic Sequence formula. There are ways to make this more efficient which you will learn in future years!

Exercise Set

"*" indicates challenge questions.

1. Extend the following geometric sequences.

   (a) 3, 6, 12, 24

   (b) 375, 75, 15, 3

   (c) 7, 21, 63, 189

   (d) 24, 12, 6, 3

   (e) 5, 20, 80, 320

   (f) $\frac{-1}{32}, \frac{1}{16}, \frac{-1}{8}, \frac{1}{4}$

2. Determine the formula for the geometric sequence $\{64, 8, 1, \frac{1}{8}, ...\}$.
   Here, $a_1 = 64$ and $c = \frac{1}{8}$ so the formula is $a_n = a_{n-1} \times \frac{1}{8}$.

3. Determine the formula for the arithmetic sequence $\{\frac{4}{5}, \frac{7}{5}, 2, \frac{13}{5}, ...\}$.
   Here, $a_1 = \frac{4}{5}$ and each term is increasing by $\frac{3}{5}$ so the formula is $a_n = \frac{4}{5} + \frac{3}{5}(n-1)$.

4. A geometric sequence is defined by $a_1 = \frac{-1}{8}$ and $a_n = 2 \times a_{n-1}$. What is a_4, the 4^{th} term in the sequence?
   Using the formula, $a_2 = a_1 \times 2 = \frac{-1}{4}$. Similarly, $a_3 = a_2 \times 2 = \frac{-1}{2}$. Finally, $a_4 = a_3 \times 2 =$ **-1**.

5. A geometric sequence is defined by $a_1 = 15$ and $a_n = -3 \times a_{n-1}$. What is a_4, the 4^{th} term in the sequence?
   Using the formula, $a_2 = a_1 \times -3 = -45$. Similarly, $a_3 = a_2 \times -3 = 135$. Finally, $a_4 = a_3 \times -3 =$ **-405**.

6. We can define an arithmetic sequence as follows:

$$\begin{cases} a_1 = 2 & \text{the first term} \\ a_n = a_{n-1} - 6 & \text{all remaining terms} \end{cases}$$

   Determine the first 5 terms in the sequence.
   $a_2 = a_1 - 6 = 2 - 6 = -4$ and doing so for the next terms we get the sequence {2, -4, -10, -16, -22, ...}.

Special Sequences

A **special sequence** is a sequence that has a unique pattern to it.

Sequence of Square Numbers

A **square number** is a number that results from multiplying a number by itself.

$$1 \times 1 = 1 \qquad 2 \times 2 = 4 \qquad 3 \times 3 = 9 \qquad etc.$$

The sequence of square numbers can be written as follows:

$$\{1, 4, 9, 16, 25, 36, 49, ...\}$$

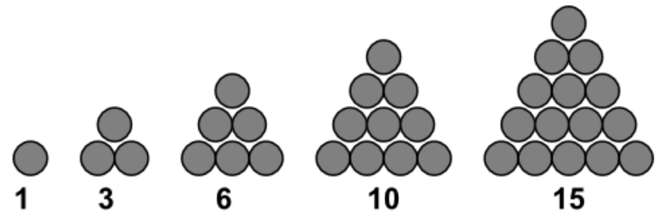
We can observe the following pattern between each term and the term number

$$\begin{array}{rcll} 1 \times 1 & = & 1 & 1^{st} \text{ term} \\ 2 \times 2 & = & 4 & 2^{nd} \text{ term} \\ 3 \times 3 & = & 9 & 3^{rd} \text{ term} \\ 4 \times 4 & = & 16 & 4^{th} \text{ term} \end{array}$$

We can continue the pattern above to conclude that for the n^{th} term, $a_n = n \times n$.

Triangular Sequence

A **triangular sequence** is a sequence that gives the number needed to form a triangle. Observe the diagram below:



The number of circles needed to make each triangle in order can be written as:

$$\{1, 3, 6, 10, 15, ...\}$$

For the pattern above, we can conclude that the n^{th} term can be written as $a_n = a_{n-1} + n$ with $a_1 = 1$.

Exercise: Following the pattern, how many circles would you need to make the next triangle?

The next triangle is the 6^{th} triangle and using the formula, $a_6 = a_5 + 6 = 15 + 6 = \mathbf{21}$ so 21 circles.

Special Sequences Continued

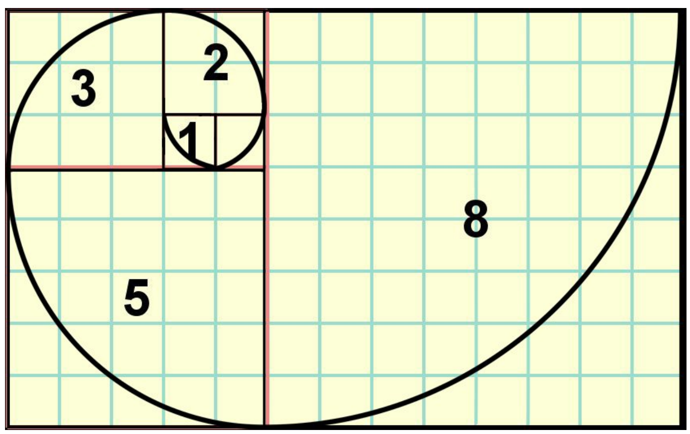
Fibonacci Sequence

The **Fibonacci sequence** is a recursive sequence where each term is generated by adding the two previous numbers in the sequence:

$$\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...\}$$

The formula for the Fibonacci sequence can be written as $a_n = a_{n-1} + a_{n-2}$ with $a_1 = 0$ and $a_2 = 1$. The Fibonacci sequence appears in the world around us. Let's see how!

Exercise: Complete the **Fibonacci spiral** in the grid below.



We can now observe how this spiral appears all around us.

Problem Set:

1. Complete the recursive formula for each sequence.

(a) {12, 10, 8, 6, ...}

$$\begin{cases} a_1 = \underline{\quad 12 \quad} \\ a_n = \underline{\quad a_{n-1} - 2 \quad} \end{cases}$$

(b) {-15, -90, -540, ...}

$$\begin{cases} h_1 = \underline{\quad -15 \quad} \\ h_n = \underline{\quad h_{n-1} \times 6 \quad} \end{cases}$$

(c) {300, 60, 12, 2.4, ...}

$$\begin{cases} d_1 = \underline{\quad 300 \quad} \\ d_n = \underline{\quad d_{n-1} \times \frac{1}{5} \quad} \end{cases}$$

2. Write the formula for a_n for each of the following sequences.

Hint: *For geometric sequences, remember to include the first term.*

(a) {-7, -2, 3, 8, 13, 18, ...} $a_n = -7 + 5(n-1)$

(b) {200, 100, 50, 25, ... } $a_n = a_{n-1} \times \frac{1}{2}, a_1 = 200$

(c) {170, 85, 0, -85, ...} $a_n = 170 - 85(n-1)$

(d) {96, 24, 6, 1.5, ...} $a_n = a_{n-1} \times \frac{1}{4}, a_1 = 96$

(e) * $\{\frac{5}{32}, 1, \frac{59}{32}, \frac{86}{32}, \ldots\}$ $a_n = \frac{5}{32} + \frac{27}{32}(n-1)$

(f) * $\{6, 4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \ldots\}$ $a_n = a_{n-1} \times \frac{2}{3}, a_1 = 6$

3. Determine the indicated term for each sequence.

(a) $a_1 = \frac{3}{16}$ and $a_n = a_{n-1} \times 4$ $\qquad a_3 = \underline{\quad 3 \quad}$

(b) $b_1 = 0$, $b_2 = 5$, and $b_n = b_{n-1} + b_{n-2}$ $\qquad b_4 = \underline{\quad 10 \quad}$

(c) $c_1 = 3$ and $c_n = c_{n-1} - 14$ $\qquad c_3 = \underline{\quad -25 \quad}$

(d) $d_1 = \frac{2}{7}$ and $d_n = d_{n-1} \times 7$ $\qquad d_5 = \underline{\quad 686 \quad}$

(e) $e_1 = 0$ and $e_n = e_{n-1} \times \frac{13}{49}$ $\qquad e_{17} = \underline{\quad 0 \quad}$

(f) $f_1 = 18$ and $f_n = f_{n-1} \times \frac{1}{6}$ $\qquad f_6 = \underline{\quad \frac{1}{432} \quad}$

4. Determine the missing terms in each sequence.

   (a) -8, -14, -20, __-26__, __-32__ ($a_n = a_{n-1} - 6$)

   (b) 189, 63, 21, __7__, __$\frac{7}{3}$__ ($a_n = a_{n-1} \times \frac{1}{3}$)

   (c) 15, 14, 29, __43__, 72, 115, 187 ($a_n = a_{n-1} + a_{n-2}$)

   (d) __81__, 64, 49, 36, __25__, 16, 9, __4__, __1__ (decreasing square numbers)

   (e) $\frac{1}{96}, \frac{1}{16}$, __$\frac{1}{4}$__ $, 1, 4$ ($a_n = a_{n-1} \times 4$)

   (f) * 0, 2, 5, __9__, 14, 20, 27 ($a_n = a_{n-1} + n$) *(This is the triangle sequence starting at 0.)*

   (g) * __$\frac{-81}{64}$__ $, \frac{27}{16}, \frac{-9}{4}, 3,$ __-4__ ($a_n = a_{n-1} \times \frac{-4}{3}$)

   (h) * 415, 257, 158, 99, 59, 40, __19__ ($a_n = a_{n-2} - a_{n-1}$)

5. Find the common difference and the n^{th} term.

   (a) {89, 78, 67, 56, 45, ...} d = -11, $a_n = 89 - 11(n-1)$

   (b) {55, 62, 69, 76, 83, ...} d = +7, $a_n = 55 + 7(n-1)$

   (c) {27, 27, 27, 27, 27, ...} d = +0, $a_n = 27$

   (d) {0.3, 0.5, 0.7, 0.9, 1.1, ...} d = +0.2, $a_n = 0.3 + 0.2(n-1)$

6. Find the common ratio and the next term.

   (a) {15, 45, 135, 405 ...} c = 3, $a_5 = a_4 \times 3 =$ **1215**

   (b) {4096, 1024, 256, 64 ...} c = $\frac{1}{4}$, $a_5 = a_4 \times \frac{1}{4} =$ **16**

   (c) {1, 1, 1, 1, 1, ...} c = 1, $a_6 = a_5 \times 1 =$ **1**

   (d) * {1, $\sqrt{5}$, 5, ...} c = $\sqrt{5}$, $a_4 = a_3 \times \sqrt{5} = 5 \times \sqrt{5} \approx$ **11.18**

7. Scott has decided to add strength training to his exercise program. His trainer suggests that he add weight lifting for 5 minutes during his routine for the first week. Each week thereafter, he is to increase the weight lifting time by 2 minutes. If Scott continues with this increase in weight lifting time, how many minutes will he be devoting to weight lifting in week 10?

   He is increasing his weight lifting time by a constant value each week. This is an *arithmetic sequence* with formula $a_n = 5 + 2(n-1)$. In week 10 he will be weigh lifting for:

   $$a_{10} = 5 + 2(10-1) = 5 + 18 = \textbf{23 minutes}$$

8. A research lab is to begin experimentation with a bacteria that doubles every 4 hours. The lab starts with 200 bacteria. How many bacteria will be present at the end of the 12^{th} hour?

   The number of bacteria is multiplied by 2 every 4 hours so this is a *geometric sequence.*

   We start with $b_1 = 200$ bacteria and after 4 hours, we have have $b_2 = 200 \times 2 = 400$ bacteria.

   In 8 hours, two 4 hour periods have passed by so we will have $b_3 = 400 \times 2 = 800$ bacteria.

   Finally, in 12 hours, three 4 hour periods have passed so $b_4 = 800 \times 2 =$ **1600 bacteria**.

9. Your father wants you to help him build a shed in the backyard. He says he will pay you \$10 for the first week and add an additional \$10.50 each week thereafter. The project will take 5 weeks. How much money will you earn, in total, if you work for the 5 weeks?

   Each week your pay is increasing by a constant value so this is an *arithmetic sequence.*

   The sequence for how much you get paid each week is {\$10, \$20.50, \$31, \$41.50, \$52}.

   To determine the total money earned we can add the money earned each week to get:

$$\$10 + \$20.50 + \$31 + \$41.50 + \$52 = \mathbf{\$155}$$

10. The summer Olympics occur every four years. Starting with 2016, in which year will the 12^{th} summer Olympics occur?

    The Olympics years increases by a constant value so this is an *arithmetic sequence.*

    The formula for the Olympic years starting at 2016 can be written as $y_n = 2016 + 4(n - 1)$.

    The 12^{th} summer Olympics occurs in $y_{12} = 2016 + 4(12 - 1) =$ **2060**.

11. The terms in the sequence {2, 7, 12, 17, 22, ..} increase by 5. The terms in the sequence {3,10, 17, 24, 31, ...} increase by 7. The number 17 occurs in **both** sequences. What is the next number that appears in **both** sequences?

    Continuing both sequences, we see that the next comon number is **52**.

12. In the sequence 32, 8, ______, ______, x, each term after the second is the average of the two terms immediately before it. What is the value of x? *(Pascal 2005, Grade 9. #10)*

    Hint: *The average of two numbers is their sum divided by 2.*

    The third term is the average of the first two terms, 32 and 8, or $\frac{1}{2}(32 + 8) = \frac{1}{2}(40) = 20$.

    The fourth term is the average of the second and third terms, or $\frac{1}{2}(8 + 20) = 14$.

    The fifth term is the average of the third and fourth terms, or $\frac{1}{2}(20 + 14) = 17$.

    Therefore, $x = \mathbf{17}$.

13. Arya plays Candy Crush every day for 7 days. Each day after the first, he plays 5 more levels than before. In total, he played 175 levels. How many levels did he play on the last day?

Solution 1:

Let n represent the number of games he played on the first day. Then he played $n+5$ games on the second day, $n+10$ on the third day, etc.

$$\begin{aligned} n+(n+5)+(n+10)+(n+15)+(n+20)+(n+25)+(n+30) &= 175 \\ 7n+105 &= 175 \\ 7n+105-105 &= 175-105 \\ 7n &= 70 \\ 7n\div 7 &= 70\div 7 \\ n &= 10 \end{aligned}$$

From the equation, he played $n+30$ games on the last day ($n=10$), so he played **40** games.

Solution 2:

Arya played games for 7 days (an odd number of days), and on each day he played an equal number of games more than the day before (5 more).

Therefore, the number of push-ups that Arya did on the middle day (day 4) is equal to the average number of games that he played each day.

Arya played 175 games in total over the 7 days, and thus on average he played $175 \div 7 = 25$ games each day.

Therefore, on day 4 Arya played 25 games, and so on day 5 he played $25+5=30$ games, on day 6 he played $30+5=35$ games and on the last day he played $35+5=$ **40 games**.

Note: We can check that $10+15+20+25+30+35+40=175$, *as required.*

14. * Penelope folds a piece of paper in half, creating two layers of paper. She folds the paper in half again, creating a total of four layers of paper. If she continues to fold the paper in half, which of the following is a possible number of layers that could be obtained?

(Cayley 2017, Grade 10, #6)

(a) 10 (b) 12 (c) 14 (d) 16 (e) 18

When Penelope folds the paper in half, the number of layers doubles.

Starting with 4 layers of paper, then after the next three folds, there are 8 and then 16 and then 32 layers of paper. Additional folds create more layers.

Of the given choices (which are all less than 32), only 16 is a possible number of layers.

15. * What is the sum of the first 2005 terms of the sequence 1, 2, 3, 4, 1, 2, 3, 4, ...?

(Fermat 2005, Grade 11, #6)

The sequence repeats every 4 terms.

How many times will the pattern 1, 2, 3, 4 occur in the first 2005 terms?

Since 2005 divided by 4 gives a quotient of 501 and a remainder of 1, then the first 2005 terms contain the pattern 1, 2, 3, 4 a total of 501 times (ending at the 2004th term).

Also, the 2005th term is a 1.

Therefore, the sum of the first 2005 terms is $501(1 + 2 + 3 + 4) + 1 = 501(10) + 1 = \mathbf{5011}$.

16. * Seven children, each with the same birthday, were born in seven consecutive years. The sum of the ages of the youngest three children is 42. What is the sum of the ages of the oldest three? *(Fermat 2005, Grade 11, #8)*

Solution 1:

Since the seven children were born in seven consecutive years, then the oldest child is 4 years older than the oldest of the three youngest children, the second oldest child is 4 years older than the second oldest of the three youngest children, and the third oldest child is 4 years older than the youngest child. Since the sum of the ages of the three youngest children is 42, then the sum of the ages of the three oldest children is $42 + 4 + 4 + 4 = \mathbf{54}$.

Solution 2:

Since the ages of the seven children are seven consecutive integers, let the ages of the youngest three children be x, $x + 1$ and $x + 2$.

Then $x + x + 1 + x + 2 = 42$ or $3x + 3 = 42$ or $x = 13$.

So the ages of the seven children are 13, 14, 15, 16, 17, 18, and 19.

Therefore, the sum of the ages of the oldest three children is $17 + 18 + 19 = \mathbf{54}$.

17. * When expressed as a decimal, $\frac{1}{7} = 0.142857142857\ldots$. Which of the following positions to the right of the decimal will be a 2? *(Gauss 2015, Grade 8, #17)*

(a) 119^{th} (b) 121^{st} (c) 123^{rd} (d) 125^{th} (e) 126^{th}

Since the number of digits that repeat is 6, then the digits 142857 begin to repeat again after 120 digits (since $120 = 6 \times 20$).

That is, the 121^{st} digit is a 1, the 122^{nd} digit is a 4, and the 123^{rd} digit is a 2.

18. ** What is the tens digit of 3^{2016}? *(Gauss 2016, Grade 8, #24)*

When two integers are multiplied together, the final two digits (the tens digit and the units digits) of the product are determined by the final two digits of each of the two numbers that are multiplied.

This is true since the place value of any digit contributes to its equal place value (and possibly also to a greater place value) in the product.

That is, the hundreds digit of each number being multiplied contributes to the hundreds digit (and possibly to digits of higher place value) in the product.

Thus, to determine the tens digit of any product, we need only consider the tens digits and the units digits of each of the two numbers that are being multiplied.

For example, to determine the final two digits of the product 1215×603 we consider the product $15 \times 03 = 45$. We may verify that the tens digit of the product $1215 \times 603 = 732645$ is indeed 4 and the units digit is indeed 5.

Since $3^5 = 243$, then the final two digits of $3^{10} = 3^5 \times 3^5 = 243 \times 243$ are given by the product $43 \times 43 = 1849$ and thus are 49.

Since the final two digits of 3^{10} are 49 and $3^{20} = 3^{10} \times 3^{10}$, then the final two digits of 3^{20} are given by $49 \times 49 = 2401$, and thus are 01.

Then $3^{40} = 3^{20}3^{20}$ ends in 01 also (since $01 \times 01 = 01$).

Further, 3^{20} multiplied by itself 100 times, or $(3^{20})^{100} = 3^{2000}$ also ends with 01.

Since 3^{10} ends with 49 and 3^5 ends with 43, then $3^{15} = 3^{10} \times 3^5$ ends with $49 \times 43 = 2107$ and thus has final two digits 07.

This tells us that $3^{16} = 3^{15} \times 3^1$ ends with $07 \times 03 = 21$.

Finally, $3^{2016} = 3^{2000} \times 3^{16}$ and thus ends in $01 \times 21 = 21$, and so the tens digit of 3^{2016} is 2.