UNIVERSITY OF WATERLOO

Faculty of Mathematics

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Centre for Education in

Mathematics and Computing

Grade 6 Math Circles

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Counting and Probability

Probability is a way to measure likelihood of an event occurring. Probability is used everyday by many people including statisticians to everyone in this classroom! Understanding it and knowing how to use it to predict outcomes can be very helpful in areas like:

Weather Forecast, Sports, Lottery (any game of chance), Medical Operations

Probabilities are often represented by fractions or decimals with values between 0 and 1:

$$\frac{3}{5} = \frac{6}{10} = 0.6 \qquad \frac{13}{26} = \frac{1}{2} = 0.5 \qquad \frac{9}{14} \approx \frac{64}{100} = 0.64 \qquad \frac{1}{3} \approx 0.33$$

Experimental Probability - an estimate of the likelihood of an event occurring based on the collection of data during experiments, direct observation, experience, or practice.

Theoretical Probability - using your knowledge about a situation, some logical reasoning, and/or a known formula to calculate the probability of an event happening

Theoretical probability is useful, because it can be calculated with an equation before any experiments need to be done. The probability of any event **A** occurring is:

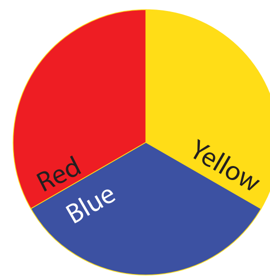
$$\text{Probability of A} = \text{P(A)} = \frac{number\ of\ ways\ \mathbf{A}\ can\ occur}{total\ number\ of\ possible\ outcomes}$$

To understand this equation better, we must introduce the following definition.

Sample Space

The **Sample Space** of a given activity is the set of all possible outcomes that can happen for that specific activity. Sample Spaces vary in size and there are several ways to find how many possibilities are included in these spaces.

Exercise: The following spinner is divided into 3 equal parts and is spun 2 times, with each spin being recorded.



1. What is the Sample Space and the size of this Sample Space?
2. Let **A** be the event that the colour yellow appears first. How many different ways can **A** occur?
3. What is the probability that event **A** will occur?

To find the probability of an event occurring, find the size of the sample space and the amount of times the event occurs within that sample space. Then, divide this number of occurrences by the size of the sample space.

Exercise Set 1 Challenge questions *.

1. A bag contains 5 red, 3 green, 2 blue and 4 yellow marbles.

   (a) If a single marble is chosen at random from the bag and its colour is recorded. What is the Sample Space and the size of this Sample Space?

   (b) If two marbles are chosen one after another from the bag and their colours are recorded in order. What is the Sample Space and the size of this Sample Space?

   (c) Two marbles are chosen one after another from the bag and colours are recorded. Let **B** be the event that the colour blue is chosen last. How many different ways can **B** occur?

2. You are given a standard deck of cards which contains 52 cards.

   (a) You choose a card and record its suit. What is the Sample Space and the size of this Sample Space? *(Let the suits be:* ***C*** *- clubs,* ***D*** *- diamonds,* ***H*** *- hearts,* ***S*** *- spades.)*

   (b) What would the Sample Space from part (a) be if you recorded the value instead of the suits? What is its size?

   (c) * You choose two cards one after another and write the suit and the value in the order that you picked the card. Let **A** be the event that both cards are Ace. How many different ways can **A** occur?

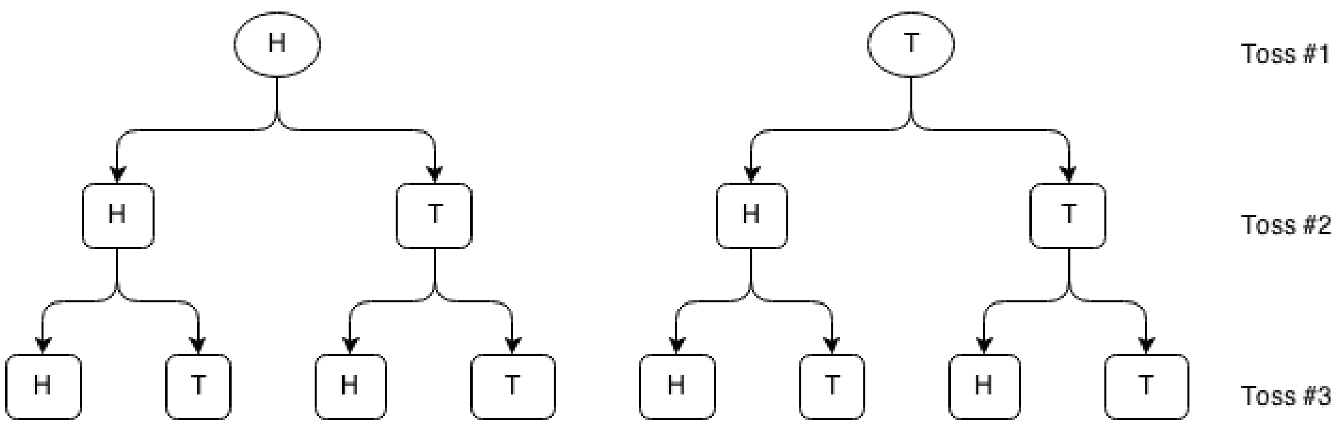
Tree Diagrams

In some cases we will find that Sample Spaces are not always easy to form. Thus we need useful and simple methods to formulate the Sample Space that also gives us the size of the Sample Space. There are many different methods to find the Sample Space and its size. One of these methods is to use a **tree diagram**.

Tree diagrams display all possible outcomes of an event. Each **branch** represents a possible outcome. Tree diagrams can be used to find the number of possible outcomes.

Example: A fair coin is flipped 3 times and the result of each flip is recorded. Recall that **fair** means to flip a head is equally as likely as flipping a tail.

Solution: In the example above, we can find the Sample Space as follows:



To form each 3 coin flip combination, we start from the top of each tree then trace down each route separately. Doing so will give us the result of the first flip, the second flip and finally the last flip. The Sample Space can be written as {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

To find the **size of the Sample Space**, we simply count the number of items in the last row. In the example above, the size of the Sample Space is **8**.

Remark: The probability of getting HHH is $\frac{1}{8}$. But notice, the probability of getting H in the first flip is $\frac{1}{2}$, the probability of getting H in the second flip is $\frac{1}{2}$, the probability of getting H in the third flip is $\frac{1}{2}$.

$$P(H) \times P(H) \times P(H) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} = P(HHH)$$

But does this always work or is this a coincidence? Keep reading to find out!

Multiple Events

Before we can answer that question, we need to learn some counting principles and new terminology.

Product Rule - Fundamental Counting Principle (FCP)

If you have to make Choice A ***AND*** *Choice B and there are m options for Choice A, n options for Choice B, then the total number of ways you can make Choice A* ***AND*** *Choice B is $m \times n$.*

Example: The ice cream truck sells ice creams in sizes regular and large in one of the 5 flavours: chocolate, vanilla, strawberry, peanut butter and mango. How many different ice creams can they make in total?

Solution: In other words, how many ice creams can they make if they have 2 sizes and 5 flavours?

$$2 \text{ sizes} \times 5 \text{ flavours} = \textbf{10 ice creams in total}$$

Sum Rule

If you have to make Choice A ***OR*** *Choice B and there are m options for Choice A, n options for Choice B, then the total number of ways you can make Choice A* ***OR*** *Choice B is $m + n$.*

Example: The Grade 5 and 6 students of a school are asked to choose whether they prefer dogs or cats. Of the 48 Grade 5 students, 36 prefer dogs and 12 prefer cats. Of the 52 Grade 6 students, 20 prefer dogs and 32 prefer cats. How many in total prefer dogs?

Solution: All students are either in Grade 5 **OR** Grade 6. The total number of students who prefer dogs are the number of Grade 5 students who prefer dogs plus the number of Grade 6 students who prefer dogs as follows:

$$36 \text{ G5 students prefer dogs} + 20 \text{ G6 students prefer dogs} = \textbf{56 total students prefer dogs}$$

Exercise:

1. How many possible outfits can be made when you have 3 shirts, 2 pants, and 5 shoes?

2. In a pet store, there are 6 puppies, 9 kittens, 4 hamsters and 7 fish. If a pet is chosen at random, how many possibilities are there of choosing a puppy or a fish?

Venn Diagrams

Let's consider the following example:

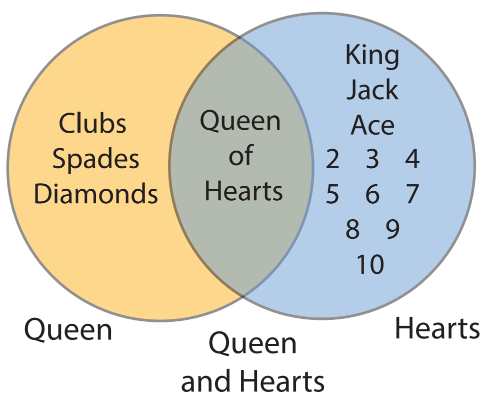
How many possibilities are there that a card randomly drawn from a deck is a queen OR a hearts card?

Using the Sum Rule doesn't give us the correct answer. The Sum Rule tells us that there would be 17 possibilities (4 queens + 13 hearts) but we are counting the Queen of Hearts twice. Once when we count the queens and once when we count the hearts. To find the correct answer we must add the choices for queen to the choices for hearts and subtract the choices that are hearts and queen to avoid double counting as follows:

$$4 \text{ queens} + 13 \text{ hearts} - 1 \text{ queen of heart} = 16 \text{ queens or hearts}$$

We can visualize this better using a Venn Diagram.

Venn Diagrams are diagrams consisting of overlapping and/or nested shapes used to show what two or more sets have and do not have in common. In the example above, the first set is the set of Queens Clubs, Diamonds, Hearts, Spades and the second set is the set of Hearts Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2. The Queen of Hearts appears in both of these sets so they have that in common. We represent this as follows:



Steps to create a Venn Diagram (2 sets):

1. Determine the sets you are comparing and list out the items in each set.

2. Draw two overlapping circles and label them.

3. Find the items that appear in both sets and write them in the section where the two circles overlap. Write the remaining items in the appropriate circle.

Intersection, Union, Mutually Exclusive

From the previous example we learned that the Sum Rule does not work for every case and so we must adjust the Sum Rule to hold for all cases. But before doing so, we must learn some new terminology:

> **Intersection**
>
> *For two events A and B, the* ***intersection*** *of A and B, denoted $(A \cap B)$, is the event that both A* ***AND*** *B happen.*

Exercise: Let event A be to flip heads last in a series of 3 flips and event B be to flip tails first in a series of 3 flips. What is the **intersection**, $(A \cap B)$ of the two events? Use a Venn Diagram to represent the two events and their intersection.

> **Union**
>
> *For two events A and B, the* ***union*** *of events A and B, denoted $(A \cup B)$, is the event that A* ***OR*** *B occurs.*

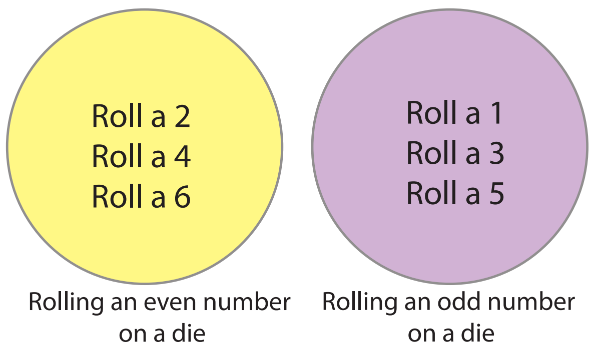
Exercise: You roll two 6-sided die. Let event A be that the sum of the two die is even and let event B be that the sum of the two die is a multiple of 5. Use a Venn Diagram to represent the two events and their intersection. What is the **union**, $(A \cup B)$ of the two events?

Exercise: In the example with the 3 coin flips above, what is the **union**?

Mutually Exclusive

*If event A **AND** event B can never occur at the same time, then they are called **mutually exclusive**. Mathematically, we would say $P(A \cap B) = 0$.*

Example: If A is the event of rolling an even number on a 6 sided die and event B is rolling an odd number on a 6 sided die, events A and B are **mutually exclusive**. The probability of their intersection, $P(A \cap B) = 0$ as you cannot roll a number that is both even and odd.



Exercise: There are 78 students in Math Circles. 35 of the students have a brother. 48 have sisters. Of the 48 who have sisters, 16 have brothers. The rest of the students have no siblings. Use a Venn Diagram to help you answer the following questions:

1. How many have no siblings?
2. How many students have siblings and don't have any sisters?
3. How many students have only a brother or only a sister but not both?

Sum Rule Revisited

Now that we have learned the needed terminology, we will revisit the Sum Product and modify it to give us the right answer.

Special Sum Rule

If you have to make Choice A ***OR*** *Choice B such that A and B are* ***mutually exclusive*** *and there are m options for Choice A, n options for Choice B, then the total number of ways you can make choices A* ***OR*** *B is m + n.* Additionally,

$$P(A \cup B) = P(A) + P(B)$$

Example: Given a bag of 3 blue balls, 5 red balls, 6 yellow balls and 6 green balls, what is the probability of picking a red ball or a green ball?

Solution: Notice how you cannot pick a ball that is both red and green so the two events are mutually exclusive so we can apply the **Special Sum Rule** as follows:

$$\text{P(green or red ball)} = \text{P(green ball)} + \text{P(red ball)} = \frac{6}{20} + \frac{5}{20} = \frac{11}{20} = 0.55$$

General Sum Rule

If you have to make Choice A ***OR*** *Choice B and there are m options for Choice A, n options for Choice B, then the total number of ways you can make choices A* ***OR*** *B is m + n - $|A \cap B|$ where $|A \cap B|$ is the number of items in both A and B. Additionally,*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Example: The numbers 1 to 20 are written on a paper and placed in a bag. What is the probability of picking a number divisible by 2 or a number divisible by 3?

Solution: Let event A be that you pick a number divisible by 2. Let event B be that you pick a number divisible by 3. Notice that if you pick the number 6 then since 6 is divisible by 2 and by 3 and so $A \cap B$ contains at least 6. So we must apply the **General Sum Rule** as follows:

$$\text{P(divisible by 2 or 3)} = \text{P(divisible by 2)} + \text{P(divisible by 3)} - \text{P(divisibile by 2 and 3)}$$
$$\text{P(divisible by 2 or 3)} = \frac{10}{20} + \frac{6}{20} - \frac{3}{20} = \frac{13}{20} = 0.65$$

Note: *There are 10 numbers divisible by 2, 6 divisble by 3 and 3 divisible by both 2 and 3.*

Independent Events

Independent

Two events A and B are said to be ***independent*** *of each other if the result in the outcome of one has no effect on the other.*

Example: A = flipping a coin. B = rolling a die. Events A and B are **independent**.

Example: There is a bag of 8 blue balls and 2 yellow balls. If event A is the first pick (without replacement) is a yellow ball and event B is the second pick is a yellow ball, event A and B are **dependent**. Event B will have a different probability depending on the result of event A. If we do pick a yellow ball on our first pick, $P(B) = \frac{1}{9} \approx 0.11$ however if we don't pick a yellow ball on our first pick, then $P(B) = \frac{2}{10} = 0.2$.

Exercise: *Which of the following pairs of events are* ***independent?*** *If a pair is not independent, explain how one event affects the other event.*

- Parking illegally and getting a parking ticket.
- Buying ten lottery tickets and winning the lottery.
- Owning a dog and growing your own herb garden.
- Stealing a car and going to jail.
- Taking a cab home and finding your favorite movie on cable.
- Studying for your geography exam and doing well in your history exam.

Product Rule Revisited

Product Rule

For two events A and B, the probability of event A ***AND*** *event B* is as follows:

$$P(A \cap B) = P(A) \times P(B)$$

Example - Independent: A = rolling a 4 on a 6-sided die. B = flipping heads when flipping a coin. What is the probability of A and B?

Solution: We know that $P(A) = \frac{1}{6}$ and $P(B) = \frac{1}{2}$. Then:

$$P(A \cap B) = P(A) \times P(B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}.$$

Example - Dependent: A goblet contains 3 red balls, 2 green balls, and 6 blue balls. We choose a ball, then another ball without putting the first one back, what is the probability that the first ball will be red and the second will be blue?

Solution: Let A = first ball will be red and B = second ball will be blue.

$P(A) = \frac{3}{11}$ as there are 3 red balls available to choose and there are 11 balls in total to choose from. However, once we pick the first ball without putting it back, there will only be 10 balls left in the goblet. Since we want A **and** B, we can assume the first ball was red and so there are still 6 blue balls left to choose from of the 10 balls left. So given A, $P(B) = \frac{6}{10}$. Then:

$$P(A \cap B) = P(A) \times P(B) = \frac{3}{11} \times \frac{6}{10} = \frac{9}{55}.$$

Exercise - Independent: *You roll two die. What is the probability that both die show 1?*

Exercise - Dependent: *A table of 5 students has 3 seniors and 2 juniors. The teacher is going to pick 2 students at random from this group to present homework solutions. Find the probability that both students selected are juniors.*

More Examples

Example: We roll a pair of dice, one after the other. We then look at the sum of the numbers shown on each die. What is the probability that we roll an odd sum greater than 5?

The first thing we always want to do is find the number of possible outcomes in the sample space. Since there are 6 possible outcomes for the first die, and for each of those outcomes we have 6 more when we roll the second die, in our sample space there are:

$$6 \times 6 = 36 \text{ possible outcomes.}$$

A good way to visualize why this is true is listing the outcomes in the shape of a square:

Die 1

Die 2		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

The odd sums greater than 5 are 7, 9 and 11.

From the table we can see that 7 appears 6 times from the combination (1,6), (6,1), (2,5), (5,2), (3,4) and (4,3).

Similarly, 9 appears 4 times from the combination (3,6), (6,3), (4,5), and (5,4) and 11 appears 2 times from the combination (5,6) and (6,5).

And so the probability of getting an odd sum greater than 5 is as follows:

$$\text{P(odd sum greater than 5)} = \text{P(7 or 9 or 11)} = \text{P}(7) + \text{P}(9) + \text{P}(11) \text{ \textit{(by the sum rule)}}$$

$$\text{P}(7) + \text{P}(9) + \text{P}(11) = \frac{6}{36} + \frac{4}{36} + \frac{2}{36} = \frac{12}{36} \approx 33\%$$

And so the probability of rolling an odd sum greater than 5 is approximately **33%**.

Problem Set

"*" indicates a challenge question.

1. What is the probability of getting a head or tail when flipping a fair coin?

2. What is the probability of rolling a 7 when you roll a 6 sided die?

3. Which of the following pairs of events are mutually exclusive?

   (a) Rolling a prime number or an even number when you roll a 6 sided die.

   ***Hint**: A **prime** is an integer greater than 1 such that its only positive divisors are 1 and itself (e.g. 2, 3, 5, 7, 11, ...).*

   (b) Rolling a multiple of 5 or rolling a non-prime when you roll a 6 sided die.

4. Which of the following pairs of events are independent?

   (a) Picking two red balls consecutively (back to back) when picking from a bag containing 10 red, 12 black and 2 white balls (with replacement).

   ***Hint**: With replacement means that after you pick the first ball and see the colour, you put it back before picking the next ball.*

   (b) Picking a diamond followed by a 3 when picking cards from a standard deck of cards (without replacement).

5. Two 6 sided die are rolled. What is the probability that:

   (a) their sum is a prime number?

   (b) their sum is not a prime number?

   (c) they both show the same number?

   ***Hint**: Use the chart showing the possible outcomes when rolling two die for part (a) and (b).*

6. You have a bag with 13 purple and 24 yellow marbles. 5 of the purple marbles are large and the rest are small. 9 of the yellow marbles are large and the rest are small. What is the probability of each of the following:

   (a) picking a yellow marble?

   (b) picking a small marble?

   (c) picking a purple and small marble?

   (d) picking a yellow or large marble?

Conditional Probaility

Conditional Probability is the likelihood of an event **B** occurring, given that event **A** has already happened. This probability is written as:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

7. Answer the following questions using conditional probability:

   (a) If P(A) = 10%, P(B) = 45%, and $P(A \cup B) = 50\%$, find $P(A \mid B)$.

   (b) Given that events **A** and **B** are mutually exclusive, without performing any calculations, find $P(A \mid B)$.

   (c) In your math class, 30% of the students passed both tests on the probability unit and 45% of them passed the first test. What percent of students that passed the first test also passed the second one?

8. A weighted coin *(it is no longer fair)* is altered so that the probability of it landing on a head for each flip is $\frac{5}{7}$. The trick coin is flipped 3 times. What is the probability that head appears on the first flip and tail appears on the last flip?

9. The Ministry of Magic is holding a lottery and has sold 2000 tickets. If Harry Potter has a $\frac{1}{16}$ chance of winning, then how many tickets did he purchase?

10. Ms. Ganji is checking for homework completion. Each student has a 60% chance of having completed their homework. Ms. Ganji selects two students at random for homework check. What is the probability that:

    (a) both students have completed their homework?

    (b) neither student has completed their homework?

    (c) only one student has completed their homework?

11. * What is the probability of hitting a bullseye on a dartboard if the bullseye has a radius of 1 cm and the board has a radius of 10 cm? Assume every dart hits the board.

    ***Hint**: Area of a Circle* $= \pi \times r^2$ *where* $\pi \approx 3.14$ *and* $r^2 = radius \times radius$

12. * In Canada, 13% of the population plays hockey, basketball and baseball. Additionally, 25% of the population plays basketball and hockey, 16% plays basketball and baseball and 21% plays hockey and baseball. If 28% of the population only play basketball and 15% play only baseball, what percent of the population plays hockey?

    ***Hint**: Use a Venn Diagram to help you visualize.*

13. * **The Monty Hall Problem**

    This is a famous math problem that deals with probability.

    If you would like a better visual or more explanation, google this problem!

    You are on a gameshow where you're asked to pick one of three closed doors. Behind two of the three doors there are goats. But behind one of them, there's a brand new car.

    (a) What is the probability of winning the car?

    (b) You've now picked a door. The gameshow host opens one of the doors you didn't pick and reveals a goat. Now there are two closed doors and one open door with a goat. The host gives you one last chance to change your door. Should you change your mind and pick the other door? Why or why not?

    ***Hint**: Does your probability of winning the car change when the host opens one of the doors? The answer to this question is the key to this problem.*