University of Waterloo
Faculty of Mathematics

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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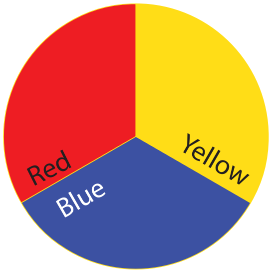
Counting and Probability Solutions

Exercise Solutions

Exercise: The following spinner is divided into 3 equal parts and is spun 2 times, with each spin being recorded.

1. What is the Sample Space and the size of this Sample Space?
2. Let **A** be the event that the colour yellow appears first. How many different ways can **A** occur?
3. What is the probability that event **A** will occur?



Solution:

1. **{RR, RB, RY, BB, BR, BY, YY, YR, YB}** where R - red, B - blue, Y - yellow. There are 9 different possible outcomes, so the size of the sample space is 9.
2. From the Sample Space above, we have the following occurrences with yellow appearing first: **{YY, YB, YR}**. Thus A can occur in 3 different ways.
3. Using our discoveries in the above questions, the probability that event **A** will occur is:

$$\text{P(A)} = \frac{\textit{number of ways } \mathbf{A} \textit{ can occur}}{\textit{total number of possible outcomes}} = \frac{3}{9} = \frac{1}{3}$$

Exercise Set 1 Challenge questions *.

1. A bag contains 5 red, 3 green, 2 blue and 4 yellow marbles.

   (a) If a single marble is chosen at random from the bag and its colour is recorded. What is the Sample Space and the size of this Sample Space?

   The Sample Space = {Red, Green, Blue, Yellow} and the size of this Sample Space is 4.

   (b) If two marbles are chosen one after another from the bag and their colours are recorded in order. What is the Sample Space and the size of this Sample Space?

   We have to consider all possible choices for the first pick and all for the second pick.

   Sample Space = {RR, RG, RB, RY, GR, GG, GB, GY, BR, BG, BB, BY, YR, YG, YB, YY} with size 16.

   (c) Two marbles are chosen one after another from the bag and colours are recorded. Let **B** be the event that the colour blue is chosen last. How many different ways can **B** occur?

   From the Sample Space above, we have the following occurences with blue being chosen last: {RB, GB, BB, YB}. Thus B can occur in 4 different ways.

2. You are given a standard deck of cards which contains 52 cards.

   (a) You choose a card and record its suit. What is the Sample Space and the size of this Sample Space? *(Let the suits be: **C** - clubs, **D** - diamonds, **H** - hearts, **S** - spades.)*

   The Sample Space = {C, D, H, S} with size 4.

   (b) What would the Sample Space from part (a) be if you recorded the value instead of the suits? What is its size?

   The Sample Space = {Ace, King, Queen, Jack, 10, 9, 8, 7, 6, 5, 4, 3, 2} with size 13.

   (c) * You choose two cards one after another and write the suit and the value in the order that you picked the card. Let **A** be the event that both cards are Ace. How many different ways can **A** occur?

   We must consider the suit of the first card and then the suit of the second card. But whatever suit appears on the first pick cannot appear on the second pick (*cannot have Clubs and Clubs*) as there is only one Ace per each suit.

   A = {CS, CD, CH, SC, SD, SH, HC, HS, HD, DC, DS, DH} thus A can occur in 12 different ways.

Exercise:

1. How many possible outfits can be made when you have 3 shirts, 2 pants, and 5 shoes?

   Here we have 3 choices to make. We must make a choice for the shirt **and** a choice for the pants **and** a choice for the shoes. This means that we have to apply the **Product Rule** as follows:

   $$3 \text{ shirts} \times 2 \text{ pants} \times 5 \text{ shoes} = 30 \text{ outfits in total}$$

2. In a pet store, there are 6 puppies, 9 kittens, 4 hamsters and 7 fish. If a pet is chosen at random, how many possibilities are there of choosing a puppy or a fish?

   The chosen pet can be a puppy **or** a fish so we can apply the **Sum Rule** as follows:

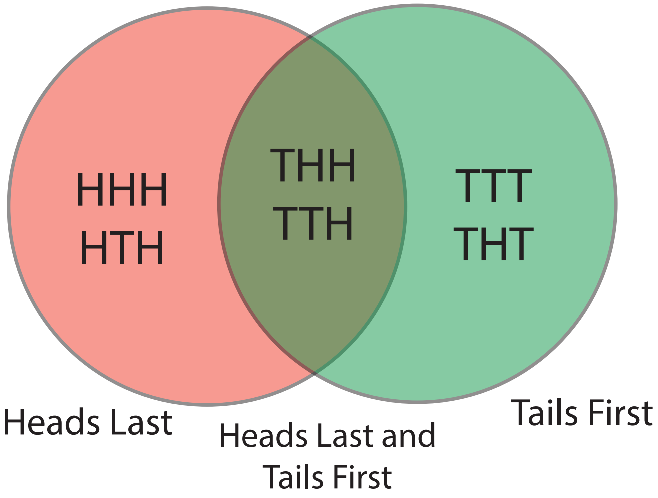
   $$6 \text{ puppies} + 7 \text{ fish} = 13 \text{ choices in total}$$

Exercise: Let event A be to flip heads last in a series of 3 flips and event B be to flip tails first in a series of 3 flips. What is the **intersection**, $(A \cap B)$ of the two events? Use a Venn Diagram to represent the two events and their intersection.

(flip head last and flip tails last) $= (A \cap B) = \{THH, TTH\}$.

Here, the Sample Space is {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

So $P(A \cap B) = \frac{2}{8} = \frac{1}{4} =$ **0.25**.



Exercise: You roll two 6-sided die. Let event A be that the sum of the two die is even and let event B be that the sum of the two die is a multiple of 5. Use a Venn Diagram to represent the two events and their intersection. What is the **union**, $(A \cup B)$ of the two events?

The first thing we always want to do is find the number of possible outcomes in the sample space. Since there are 6 possible outcomes for the first die, and for each of those outcomes we have 6 more when we roll the second die, in our sample space there are:

$$6 \times 6 = 36 \text{ possible outcomes.}$$

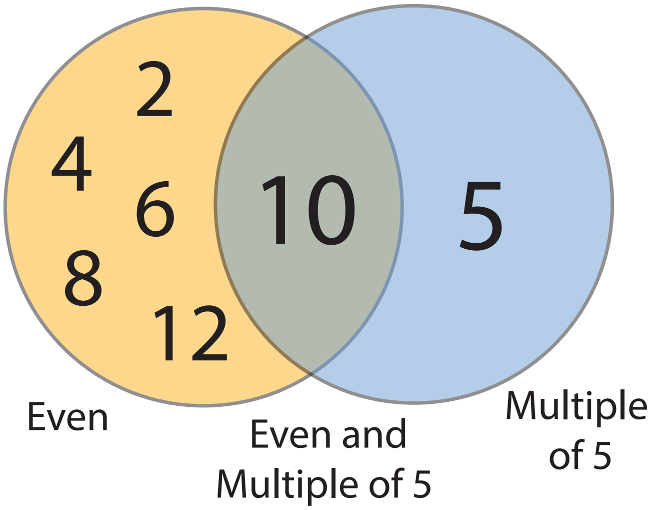
A good way to visualize why this is true is listing the outcomes in the shape of a square:

Die 1

		1	2	3	4	5	6
	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
Die 2	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

The even sums are 2, 4, 6, 8, 10, 12 and the sums that are multiples of 5 are 5 and 10. Notice that 10 appears in both.

$(A \cup B) = 2, 4, 6, 8, 10, 12, 5$



Exercise: In the example with the 3 coin flips above, what is the **union**?

The union, $(A \cup B) = \{$HHH, HTH, THH, TTH, TTT, THT$\}$.

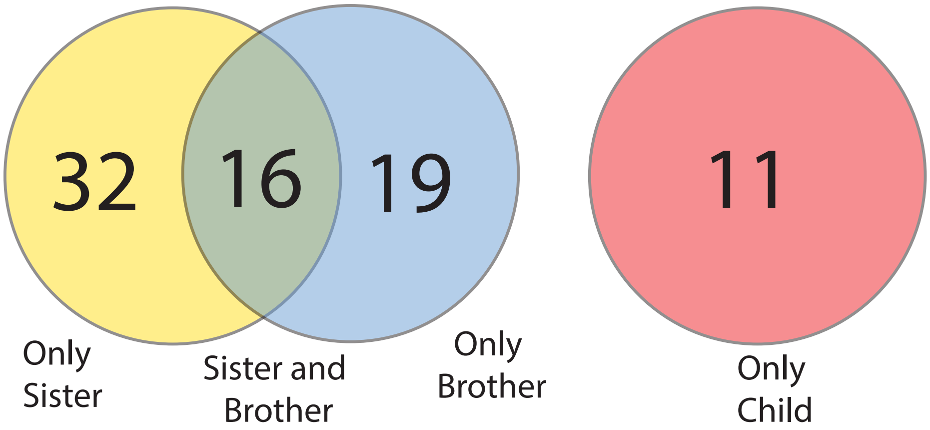
Exercise: There are 78 students in Math Circles. 35 of the students have a brother. 48 have sisters. Of the 48 who have sisters, 16 have brothers. The rest of the students are only children. Use a Venn Diagram to help you answer the following questions:

We know that of the 78 students, each student either has a brother, a sister, both or neither.

We are given that 35 have brothers, 48 have sisters and 16 have a sister and a brother so both. This means that only $35 - 16 = 19$ students have **only** a brother and $48 - 16 = 32$ have **only** a sister.

To find the number of students who are an only child, we subtract the number of students who have a sister or a brother or both from the total to get: $78 - 19 - 32 - 16 = 11$ students have no siblings.

Now we can form the following Venn Diagram to answer the questions:



1. How many have no siblings?

   11 students are only children.

2. How many students have siblings and don't have any sisters?

   Those who have siblings and don't have any sisters are those who have only brothers and **19** students have only brothers.

3. How many students have only a brother or only a sister but not both?

   From the Venn Diagram we can see that 32 have only a sister and 19 have only a brother so **51** students have only a brother or only a sister and not both.

Exercise: *Which of the following pairs of events are **independent?** If a pair is not independent, explain how one event affects the other event.*

- Parking illegally and getting a parking ticket.

  Getting a parking ticket is dependent on parking illegally and parking illegally increases the chance of getting a ticket.

- Buying ten lottery tickets and winning the lottery.

  Winning the lottery is dependent on buying tickets and the more tickets bough, the higher the chance of winning.

- Owning a dog and growing your own herb garden.

  The two events are independent, one does not affect the other.

- Stealing a car and going to jail.

  Committing a crime such as stealing a car increases the chance of going to jail so going to jail depends on stealing a car.

- Taking a cab home and finding your favorite movie on cable.

  The two events are independent, one does not affect the other.

- Studying for your geography exam and doing well in your history exam.

  The two events are independent, one does not affect the other.

Exercise - Independent: *You roll two die. What is the probability that both die show 1?*

Let event A be that you roll a 1 on the first die and event B be that you roll a 1 on the second die. The outcome of one die does not affect the outcome of the other die so the two events are independent. P(A) $= \frac{1}{6}$ and P(B) $= \frac{1}{6}$ so by **Product Rule**:

$$P(A \cap B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Exercise - Dependent: *A table of 5 students has 3 seniors and 2 juniors. The teacher is going to pick 2 students at random from this group to present homework solutions. Find the probability that both students selected are juniors.*

Let event A be that the first student chosen is a junior and event B be that the second student chosen is a junior. Clearly P(A) = $\frac{2}{5}$ since there are 2 possible juniors to choose from and 5 students in total. However, when picking the second student, there are only 4 choices to choose from and only 1 of the remaining students is a junior so given A, P(B)= $\frac{1}{4}$. Then:

$$P(A \cap B) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10} = 0.1$$

Problem Set Solutions

"*" indicates challenge question

1. What is the probability of getting a head or tail when flipping a fair coin?

   Here we have the probability of a head *or* tail so we must apply the Sum Rule and since getting a tail or a head is *mutually exclusive*, we can apply the Special Sum Rule.
   We know that $P(head) = \frac{1}{2}$ and $P(tail) = \frac{1}{2}$ and so:

   $$P(head\ or\ tail) = P(head) + P(tail) = \frac{1}{2} + \frac{1}{2} = 1 = 100\%$$

   This is because there are only 2 possible outcomes when flipping a coin. It is **certain** that you'll get a head or a tail if you flip a coin. A probability of 1 indicates that the event is certain or that it happens 100% of the time.

2. What is the probability of rolling a 7 when you roll a 6 sided die?
   $P(rolling\ a\ 7) = 0$
   This is because you cannot roll a 7 when rolling a 6 sided die. It is **impossible**. The probability of an impossible event is 0 as it can never happen or in other words, it happens 0% of the time.

3. Which of the following pairs of events are mutually exclusive?

   (a) Rolling a prime number or an even number when you roll a 6 sided die.

   Hint: *A* ***prime*** *number is a number that is only divisible by 1 and itself (e.g. 2, 3, 5, 7, 11, ...).*

   Let event A represent rolling a prime number and let event B represent rolling an even number.

   Clearly 2 is prime as it is only divisble by 1 and itself and 2 is even. So $A \cap B$ contains at least 2 and so $P(A \cap B) \neq 0$ and so the pair of events are **not** mutually exclusive.

   (b) Rolling a multiple of 5 or rolling a non-prime when you roll a 6 sided die.

   Let event A be to roll a multiple of 5 and event B be to roll a non-prime. Event A = {5} and event B = {1,4,6}. Clearly $A \cap B$ is empty as there is no number that is in A and in B. So the pair of events are mutually exclusive.

4. Which of the following pairs of events are independent?

   (a) Picking two red balls consecutively (back to back) when picking from a bag containing 10 red, 12 black and 2 white balls (with replacement).

   Hint: *With replacement means that after you pick the first ball and see the colour, you put it back before picking the next ball.*

   Independent.

   Let event A be that you pick a red ball on your first pick. Let event B be that you pick a red ball on your second pick.

   Event A will always have probability $\frac{10}{24}$. Since we are given that we pick with replacement, that means that once we pick the first ball and note down the colour, we put it back in the bag. This means that the number of red balls in the bag remains 10 and the total number of balls in the bag remains 24 and so the probability of choosing a red ball in our second pick does not change and remains $\frac{10}{24}$. So the probability of B does **not** depend on A and so the two events are independent.

(b) Picking a diamond followed by a 3 when picking cards from a standard deck of cards (without replacement). **Dependent.**

This is different from part (a) since we don't replace the card after we pick it. Let event A be that we pick a diamond card in our first pick and event B be that we pick a 3 card in our second pick.

Now suppose in our first pick, we pick the 3 of diamonds (this has probability $\frac{1}{52}$). Since we don't replace it, for our second pick, we only have 3 available choices for 3 (3 of spades, 3 of clubs and 3 of hearts) out of 51 cards and so $P(B) = \frac{3}{51}$. On the other hand, suppose we pick a diamonds card that is **not** a 3 (this has probability $\frac{12}{52}$). Now for our second pick we have 4 available choices for 3 and so $P(B) = \frac{4}{51}$. So the outcome of B depends on the outcome of A and so they are **dependent** events.

5. Two 6 sided die are rolled. What is the probability that:

We will be using the following table to answer the this question.

		Die 1					
		1	2	3	4	5	6
Die 2	1	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

(a) their sum is a prime number?

Looking at all the possible sums, we note the prime sums are 2, 3, 5, 7, and 11. So we are looking at the outcomes that give us a sum of 2 or 3 or 5 or 7 or 11. These events are mutually exclusive as you cannot get a sum that is 5 and 11 or a sum that is 3 and 11 and so on. So we can apply the Special Sum Rule as follows:

$$\text{P(odd prime)} = \text{P}(2) + \text{P}(3) + \text{P}(5) + \text{P}(7) + \text{P}(11)$$

$$\text{P(odd prime)} = \frac{1}{36} + \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{2}{36} = \frac{15}{36} \approx 0.42 = 42\%$$

(b) their sum is not a prime number?

One way to solve this is to count all the possible sums that are not prime and use Sum Rule to find the total probability.

However, we know that number is either prime or it isn't prime (the event of being prime and the event of not being prime are mutually exclusive). So of all the possible sums, each sum is either prime or it isn't prime. Since we know that 42% of the possible sums are prime, then we can find the sums that are not prime as follows:

$$\text{P(all possible sums) - P(prime sums)} = \text{P(sums that are not prime)}$$
$$\text{P(sums that are not prime)} = 1 \text{ - } \frac{15}{36} = \frac{21}{36} \approx 0.58 = 58\%$$

(c) they both show the same number? For this to happen, both die need to be 1 or both be 2 or both be 3, ..., or both be 6. These 6 events are all mutually exclusive so we can apply the Special Sum Rule as follows:

$$\text{P(both show the same number)} = \text{P(both 1)} + \text{P(both 2)} + \text{P(both 3)} + \text{P(both 4)} + \text{P(both 5)} + \text{P(both 6)}$$

Now the probability of both die showing the number 1 means that die 1 needs to be 1 **and** die 2 needs to be 1. This is the Product Rule as follows:

$$\text{P(both show 1)} = \text{P(die 1 is 1)} \times \text{P(die 2 is 1)} = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Applying this to the other possibilities 2, 3, 4, 5, and 6, we get:

$$\text{P(both show the same number)} = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{6}{36} \approx 0.17 = 17\%$$

Hint: *Use the chart showing the possible outcomes when rolling two die for part (a) and (b).*

6. You have a bag with 13 purple and 24 yellow marbles. 5 of the purple marbles are large and the rest are small. 9 of the yellow marbles are large and the rest are small. What is the probability of each of the following:

   (a) picking a yellow marble?

   There are 13 purple marbles and 24 yellow marbles so there are 37 marbles in total.

   $$P(yellow\ marble) = \frac{24}{37}$$

   (b) picking a small marble?

   There are 8 small purple marbles and 15 small yellow marbles so there are 23 small marbles in total.

   $$P(small\ marble) = \frac{23}{37}$$

   (c) picking a purple and small marble?

   There are 13 purple marbles and 8 of them are small.

   $$P(purple\ and\ small\ marble) = \frac{8}{37}$$

   (d) picking a yellow or large marble?

   There are 24 yellow marbles and 14 large marbles and 9 marbles that are both yellow and large. So we apply the General Sum Rule to get:

   $$P(yellow\ or\ large\ marble) = \frac{24}{37} + \frac{14}{37} - \frac{9}{37} = \frac{29}{37}$$

Conditional Probaility

Conditional Probability is the likelihood of an event **B** occurring, given that event **A** has already happened. This probability is written as:

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}$$

7. Answer the following questions using conditional probability:

   (a) If P(A) = 10%, P(B) = 45%, and $P(A \cup B) = 50\%$, find $P(A \mid B)$.

   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ so 50% = 10% + 45% - $P(A \cap B)$ which gives $P(A \cap B) = 5\%$.

   Then $P(A \mid B) = \frac{5\%}{45\%} \approx 11\%$.

(b) Given that events **A** and **B** are mutually exclusive, without performing any calculations, find $P(A \mid B)$.

Given that events A and B are mututally exclusive, by definition, $P(A \cap B) = 0$ and so $P(A \mid B) = \dfrac{0}{P(B)} = 0$.

(c) In your math class, 30% of the students passed both tests on the probability unit and 45% of them passed the first test. What percent of students that passed the first test also passed the second one?

Let A be the event that they passed the first test and B be the event that they passed the second test. We are given that they passed the first test which means that P(A) = 45%. We now want to know that given A, what was the probability of B and so we use conditional probability:

$$P(pass\ \ the\ \ second\ \ test\ \ given\ \ that\ \ they\ \ passed\ \ the\ \ first) = \frac{P(passed\ \ both)}{P(passed\ \ the\ \ first)}$$

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{0.30}{0.45} \approx 0.67 = 67\%$$

8. A weighted coin *(it is no longer fair)* is altered so that the probability of it landing on a head for each flip is $\frac{5}{7}$. The trick coin is flipped 3 times. What is the probability that head appears on the first flip and tail appears on the last flip?

First, since P(head) = $\frac{5}{7}$ then we must have $P(tail) = \frac{2}{5}$.

We want the first flip to be heads and the last to be tails but the second flip is not specified so it can be a head **or** a tail so we use Sum Rule to get P(HHT or HTT) = P(HHT) + P(HTT). We must find P(HHT) and P(HTT) using Product Rule as follows:

$$P(HHT) + P(HTT) = P(H) \times P(H) \times P(T) + P(H) \times P(T) \times P(T)$$

$$P(HHTorHTT) = \frac{50}{343} + \frac{20}{343} = \frac{70}{343} \approx 0.20 = 20\%$$

9. The Ministry of Magic is holding a lottery and has sold 2000 tickets. If Harry Potter has a $\frac{1}{16}$ chance of winning, then how many tickets did he purchase?

$$P(winning) = \frac{Number\ \ of\ \ tickets}{Total\ \ number\ \ of\ \ tickets} = \frac{Number\ \ of\ \ tickets}{2000} = \frac{1}{16}$$

We can use $16 \times 125 = 2000$ and the equality of fractions to find that he bought 125 tickets.

10. Ms. Ganji is checking for homework completion. Each student has a 60% chance of having completed their homework. Ms. Ganji selects two students at random for homework check. What is the probability that:

   (a) both students have completed their homework?

   P(student has completed their homework) = 60%

   P(student has not completed their homework) = 100% - 60% = 40%

$$\begin{aligned} P(\text{Both completed homework}) &= P(\text{Student 1 complete} \cap \text{Student 2 complete}) \\ &= P(\text{Student 1 complete}) \times P(\text{Student 1 complete}) \\ &= 0.6 \times 0.6 \\ &= 0.36 \\ &= 36\% \end{aligned}$$

   (b) neither student has completed their homework?

$$\begin{aligned} P(\text{Both incomplete homework}) &= P(\text{Student 1 incomplete} \cap \text{Student 2 incomplete}) \\ &= P(\text{Student 1 incomplete}) \times P(\text{Student 1 incomplete}) \\ &= 0.4 \times 0.4 \\ &= 0.16 \\ &= 16\% \end{aligned}$$

   (c) only one student has completed their homework?

$$\begin{aligned} P(\text{Only completed homework}) &= P(\text{Only student 1 complete} \cup \text{Only student 2 complete}) \\ &= P(\text{Only student 1 complete}) + P(\text{Only student 1 complete}) \\ &= P(\text{Student 1 complete}) \cap P(\text{Student 2 incomplete}) \\ &\quad + P(\text{Student 1 incomplete}) \cap (\text{Student 2 complete}) \\ &= (0.6 \times 0.4) + (0.4 \times 0.6) \\ &= 0.48 \\ &= 48\% \end{aligned}$$

11. * What is the probability of hitting a bullseye on a dartboard if the bullseye has a radius of 1cm and the board has a radius of 10cm?

    ***Hint**: Area of a Circle* $= \pi \times r^2$ *where* $\pi \approx 3.14$ *and* $r^2 = radius \times radius$

    The probability of hitting a bullseye can be found as follows:

    $$P(Hitting\ a\ bullseye) = \frac{Area\ of\ the\ bullseye}{Total\ available\ area\ on\ the\ board}$$

    Using the hint, the area of the bullseye $A_{bullseye} = \pi \times 1cm \times 1cm = \pi cm^2$ and the area of the board $A_{board} = \pi \times 10cm \times 10cm = 100\pi cm^2$.

    Then the probability of hitting a bullseye is:

    $$P(Hitting\ a\ bullseye) = \frac{\pi\ cm^2}{100\ \pi\ cm^2} = \frac{1}{100} = 0.01 = 1\%$$

12. * In Canada, 13% of the population plays hockey, basketball and baseball. Additionally, 25% of the population plays basketball and hockey, 16% plays basketball and baseball and 21% plays hockey and baseball. If 28% of the population only play basketball and 15% play only baseball, what percent of the population plays hockey?

    ***Hint**: Use a Venn Diagram to help you visualize.*

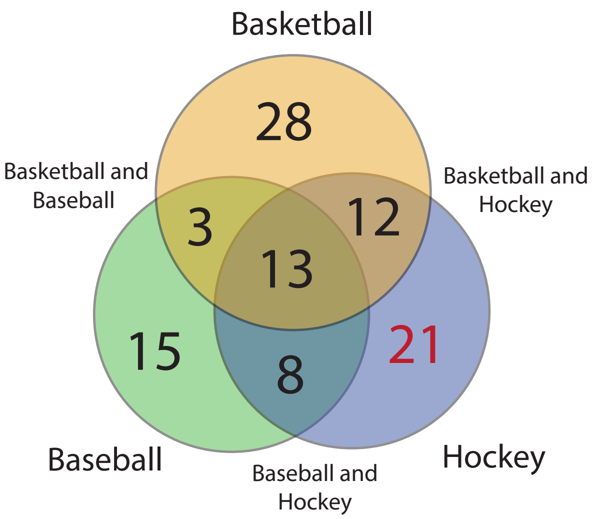
    We use a 3 circle diagram such as the one pictured below to complete this question.

    We are given that 13% of the population plays all 3 sports so we can fill the intersection of all 3 circles with 13. We are then given that 25% of the population plays basketball and hockey but we must remember that 13% of those are the ones that play all 3 sports so only 12% play only basketball and hockey and so we fill the intersection of of the basketball circle and hockey circle with 12. In a similar way we find 3 and 8. Lastly, 28% of the population plays only basketball and no other sports so we fill the remaining part of the basketball circle with 28 and we do the same for the baseball circle to get 15. Now we have all numbers except one but we know that the entire population plays at least one sport so the total of these numbers must add up to 100%. So 21% of the population plays only hockey since:

    P(Play **only** hockey) = 100% - 28% - 3% - 13% - 12% - 15% - 8% = 21%

    But we want the percent of the population that plays hockey which is:

    P(Play hockey) = 12% + 13% + 8% + 21% = 54%.



13. * **The Monty Hall Problem**

This is a famous math problem that deals with probability.

If you would like a better visual or more explanation, google this problem!

You are on a gameshow where you're asked to pick one of three closed doors. Behind two of the three doors there are goats. But behind one of them, there's a brand new car.

(a) What is the probability of winning the car?

You want to find the probability of winning the car or P(car). You know that there are 3 possible outcomes in this experiment of picking a door at random.

Since youve got 2 goats and 1 car, theres only 1 way of picking the door with the car behind it. This means that:

$$P(car) = \frac{Number\ of\ ways\ of\ picking\ the\ car}{Total\ number\ of\ possible\ outcomes} = \frac{1}{3}$$

So the probability of picking the car is $\frac{1}{3}$.

(b) You've now picked a door. The gameshow host opens one of the doors you didn't pick and reveals a goat. Now there are two closed doors and one open door with a goat. The host gives you one last chance to change your door. Should you change your mind and pick the other door? Why or why not?

Hint: *Does your probability of winning the car change when the host opens one of the doors? The answer to this question is the key to this problem.*

You have now picked a door. The gameshow host opens one of the doors you didn't pick to reveal a goat. You now have two closed doors, one of which is the one you picked. You're given a chance to change the door you picked.

The question asks you whether or not you should change your choice.

To answer this question, we need to understand what happens to P(car) when the host opens one of the doors with a goat behind it.

A good number of people would say that it doesn't matter if you change your choice or not since $P(car)$ increases to $\frac{1}{2}$ or 50%. According to them, this is because the number of possible outcomes is now reduced to 2 instead of 3.

Although this answer sounds right, it's actually **wrong**.

The reason has to do with the fact that you made the choice when all three doors were closed (i.e. there were 3 possible outcomes).

When you first pick your door, $P(car) = \frac{1}{3}$. This means that the car has a $\frac{2}{3}$ probability of being behind one of the doors you **didn't** pick.

Now, when the host opens one of the doors you didn't pick and reveals a goat, then there's a $\frac{2}{3}$ probability that the car is behind the other closed door.

Since there's a 2 in 3 chance that the car is behind the remaining closed door (that you didn't pick), the answer is that **you should change your choice every single time**.

To see how this solution works, it might be helpful to think of a game with 100 doors, 99 goats and 1 car. As before, you pick a door at random. There's a $\frac{1}{100}$ or 1% chance that the car is behind the door you picked.

If the host opens 98 of the remaining doors and reveals goats behind each and every one of them, then there's a $\frac{99}{100}$ chance that the car is behind the last closed door that you didn't pick. So if you're given a chance to change your choice, you should in order to increase your chances of winning.