Intermediate Math Circles
February 12, 2020
Contest Prep: Equations and Algebra

First: systems of linear equations
Second: systems of non-linear equations

Examples:

- $x - 2y - z = 1$ (linear)
- $x^2 + y^2 = 4$ (non-linear)
- $\sqrt{x} - \sqrt{y} = 1$ (non-linear)
- $x + xy = 3$ (non-linear)
- $4^x = 128$ (non-linear)

Systems of Linear Equations:

Two equations, two unknowns:

Example 1: Solve the following system of equations for x and y:

$$\begin{aligned} 2x + y &= 5 \\ x - 3y &= 6 \end{aligned}$$

We can use **substitution** or **elimination**:

Substitution:
Use one equation to isolate for one of the variables, then substitute into the other equation.

$$\begin{aligned} 2x + y &= 5 \qquad (1) \\ x - 3y &= 6 \qquad (2) \end{aligned}$$

Choose an equation and solve for y in terms of x or x in terms of y: From (1): $y = 5 - 2x$
Substitute this expression for y into equation (2) and solve for x:

$$\begin{aligned} x - 3y &= 6 \\ x - 3(5 - 2x) &= 6 \\ x - 15 + 6x &= 6 \\ 7x &= 21 \\ x &= 3 \end{aligned}$$

Substitute $x = 3$ into $y = 5 - 2x$ and solve for y:

$$\begin{aligned} y &= 5 - 2x \\ &= 5 - 2(3) \\ &= -1 \end{aligned}$$

Therefore, the solution to the linear system is $(x, y) = (3, -1)$

Elimination:
Solve

$$\begin{aligned} 2x + y &= 5 \qquad (1) \\ x - 3y &= 6 \qquad (2) \end{aligned}$$

If we multiply the second equation by the constant 2, the solutions to this system will not change. When we do so, we get the system

$$\begin{aligned} 2x + y &= 5 \\ 2x - 6y &= 12 \end{aligned}$$

Since $2x$ occurs in both equations, we can *eliminate* x by subtracting the second equation from the first:

$$\begin{aligned} 2x + y &= 5 \\ 2x - 6y &= 12 \\ \hline 7y &= -7 \\ y &= -1 \end{aligned}$$

Now substitute the value of y into either equation (1) or equation (2) to solve for x:

$$\begin{aligned} \text{Using (2),} \quad x - 3(-1) &= 6 \\ x &= 6 - 3 \\ x &= 3 \end{aligned}$$

When performing operations on equations, we can only perform operations that do not change the solution set. The allowed operations are:

1. multiply any equation by any non-zero constant
2. add any two equations
3. subtract any two equations

The techniques used to solve systems of two linear equations with two unknowns can be used to solve systems of linear equations with more equations and more unknowns:

Example 2:
Three different numbers are chosen such that when each of the numbers is added to the average of the remaining two, the numbers 65, 69, and 76 result. What is the average of the three original numbers? (2001 Cayley #19)

Let the numbers be a, b, and c.
The first piece of information gives $a + \frac{b+c}{2} = 65$, which gives $2a + b + c = 130$ (1).
The second piece of information gives $b + \frac{a+c}{2} = 69$, which gives $a + 2b + c = 138$ (2).
The third piece of information gives $c + \frac{a+b}{2} = 76$, which gives $a + b + 2c = 152$ (3).

Adding (1), (2), and (3) gives:

$$4a + 4b + 4c = 130 + 138 + 152$$

and so

$$4(a + b + c) = 420$$

Therefore, $a + b + c = 105$ and the average is $\dfrac{a+b+c}{3} = \dfrac{105}{3} = 35$.

Note: we could go on to solve for a, b, and c and find that $a = 25$, $b = 33$, $c = 47$, but this extra work is not necessary!

Systems of Non-linear equations:

Use some combination of

- knowledge of how to solve systems of linear equations (substitution, elimination)
- factoring and regrouping
- creativity (add/subtract equations, multiply equations, etc.)

Some useful things to know about multiplying monomials and binomials:

$2(x+y) = (x+y)+(x+y) = 2x+2y$
$a(x+y) = ax+ay$
What if we replace a with $2x+y$?
$(2x+y)(x+y) = (2x+y)x+(2x+y)y = 2x^2+yx+2xy+y^2 = 2x^2+3xy+y^2$

In general,
$(a+b)(x+y) = ax+ay+bx+by$

A special case:
$(x-y)(x+y) = x^2-y^2$ **"difference of squares" formula**

Some useful things to know about solving equations:

If $A \times B = 0$, then either $A = 0$ or $B = 0$.

If $ax+b=0$ and $a \neq 0$, then $x = -\frac{b}{a}$.

If $ax^2+bx+c=0$ and $a \neq 0$, then $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$ **the "quadratic formula"**

Sometimes recalling exponent laws can be helpful:

For any real numbers a, b, x with $x \neq 0$:

$$(x^a)(x^b) = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{ab}$$

$$x^{-a} = \frac{1}{x^a}$$

Example 3: Solve the following system of equations

$$\sqrt{x}+\sqrt{y}=36$$
$$x-y=144$$

From the "difference of squares" formula, we know that

$$\begin{aligned}(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})&=x-y\\ 36(\sqrt{x}-\sqrt{y})&=144\\ \sqrt{x}-\sqrt{y}&=\frac{144}{36}=4\end{aligned}$$

Therefore,

$$\begin{array}{r}\sqrt{x}+\sqrt{y}=36\\ \sqrt{x}-\sqrt{y}=4\\ \hline 2\sqrt{x}\qquad\quad=40\end{array}$$

So, $\sqrt{x}=20$, and so $x=400$. Therefore, $y=400-144=256$.

Example 4: Determine all (x,y) pairs where x and y are positive integers and $x+xy=391$ (2012 Cayley #21).

Factoring $x+xy=391$, we get $x(1+y)=391$.
Since x and y are positive integers, x and $1+y$ are positive integers that multiply to 391.
$391=17(23)$.
Since 17 and 23 are both prime, then if 391 is written as the product of two positive integers, it must be

$$1\times 391 \quad\text{or}\quad 17\times 23 \quad\text{or}\quad 23\times 17 \quad\text{or}\quad 391\times 1$$

Matching x and $1+y$ to these possible factors, we obtain $(x,y)=(1,390)$ or $(17,22)$ or $(23,16)$, or $(391,0)$.
Since y is a positive integer, the fourth pair is not possible.
Therefore, $(x,y)=(1,390)$ or $(17,22)$ or $(23,16)$.