Intermediate Math Circles
February 12, 2020
Contest Prep: Equations and Algebra

Problem Set

1. The symbol $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ equals $ad - bc$. If $\begin{array}{|c|c|}\hline x-1 & 2 \\ \hline 3 & -5 \\ \hline\end{array} = 9$, determine the value of x (2004 Pascal #18).

$$\begin{aligned}(x-1)(-5) - 2(3) &= 9\\ -5x + 5 - 6 &= 9\\ -5x - 1 &= 9\\ -5x &= 10\\ x &= -2\end{aligned}$$

2. In the diagram, each of the five boxes is to contain a number. For every box other than the two on the ends, the number in that box must be the average of the number in the box to the left of it and the number in the box to the right of it. Determine the values of x, y and z (2010 Pascal #21).

8	x	y	26	z

$$26 = \tfrac{y+2}{2} \implies y + z = 52 \quad (1)$$

$$y = \tfrac{x+26}{2} \implies 2y = x + 26 \quad (2)$$

$$x = \tfrac{8+y}{2} \implies 2x = 8 + y \quad (3)$$

From (2): $-x + 2y = 26$
From (3): $2x - y = 8$ $\implies$

$$\begin{aligned}-x + 2y &= 26\\ 4x - 2y &= 16\\ \hline 3x &= 42\\ x &= 14\end{aligned}$$

Therefore, $y = 2x - 8 = 2(14) - 8 = 20$

and $z = 52 - y = 32$.

3. In each row of the table, the sum of the first two numbers equals the third number. Also, in each column of the table, the sum of the first two numbers equals the third number. What is the sum of the nine numbers in the table? (2006 Pascal #21)

m	4	$m+4$
8	n	$8+n$
$m+8$	$4+n$	6

From column 3:
$m+4+8+n=6 \implies m+n=-6$

The sum of the nine numbers in the table is:

$$\begin{aligned} 2(m+8)+2(4+n)+(m+4)+(8+n)+6 &= 3m+3n+16+8+4+8+6 \\ &= 3(m+n)+42 \\ &= 3(-6)+42 \\ &= 24 \end{aligned}$$

4. A bank teller has some stacks of bills. The total value of the bills in each stack is \$1000. Every stack contains at least one \$20 bill, at least one \$50 bill, and no other types of bills. If no two stacks have the same number of \$20 bills, what is the maximum possible number of stacks that the teller could have? (2015 Cayley #19)

Consider a stack of bills worth \$1000 that includes x \$20 bills and y \$50 bills.
Then we have $20x+50y=1000$, or $2x+5y=100$.
Determining the number of possible stacks the teller could have is equivalent to determining the number of pairs (x,y) with $x \geq 1$ and $y \geq 1$ and $2x+5y=100$.
Since $2x+5y=100$, $5y=100-2x$, and so y must be even.
Therefore, the possible values for y are $2,4,6,8,10,12,14,16,18$. Once $y \geq 20$, we have $5y \geq 100$ and so $x \leq 0$.
Each of these values of y gives a pair (x,y) that satisfies the equation $2x+5y=100$:

$$(x,y)=(45,2),(40,4),(35,6),(30,8),(25,10),(20,12),(15,14),(10,16),(5,18)$$

Therefore, the maximum number of stacks that the teller could have is 9.

5. There are four unequal, positive integers a, b, c and N such that $N = 5a + 3b + 5c$. It is also true that $N = 4a + 5b + 4c$ and N is between 131 and 150. What is the value of $a + b + c$? (1998 Cayley #23)

$$\begin{array}{rl} N &= 5a + 3b + 5c \\ (-)N &= 4a + 5b + 4c \\ \hline 0 &= a - 2b + c \implies a + c = 2b \end{array}$$

$$\begin{aligned} \therefore N &= 4(a+c) + 5b \\ &= 8b + 5b \\ &= 13b\,, \text{ where } b \text{ is an integer} \end{aligned}$$

Multiples of 13 between 131 and 150:

$$\begin{aligned} 13 \times 10 &= 130 \\ 13 \times 11 &= 143 \leftarrow \text{the only one} \\ 13 \times 12 &= 156 \end{aligned}$$

$$\therefore N = 143 \implies 13b = 143 \implies b = 11$$

$$\therefore a + c = 2b = 22$$

$$\therefore a + b + c = 33$$

6. If x and y are positive integers with $xy = 6$, what is the sum of all the possible values of $\dfrac{2^{x+y}}{2^{x-y}}$? (2019 Cayley #19)

$\dfrac{2^{x+y}}{2^{x-y}} = 2^{x+y-(x-y)} = 2^{2y} = 4^y$
Since $xy = 6$ with x and y positive integers, then the possible values for y are 1, 2, 3, and 6.
Therefore, the possible values for 4^y are $4^1, 4^2, 4^3$, and 4^6.
We have $4^1 + 4^2 + 4^3 + 4^6 = 4 + 16 + 64 + 4096 = 4180$.

7. Suppose that x and y are positive numbers with

$$xy = \frac{1}{9}$$
$$x(y+1) = \frac{7}{9}$$
$$y(x+1) = \frac{5}{18}$$

What is the value of $(x+1)(y+1)$? (2011 Cayley #21)

Solution 1

Multiplying the 2^{nd} and 3^{rd} equations together we get:

$xy(x+1)(y+1) = \frac{7}{9} \cdot \frac{5}{18} = \frac{35}{162}$

but we know that $xy = \frac{1}{9}$, so:

$\frac{1}{9}(x+1)(y+1) = \frac{35}{162}$

$\therefore (x+1)(y+1) = \frac{35}{9 \cdot 18} \cdot 9 = \frac{35}{18}$

Solution 2

From 2^{nd} equation:

$xy + x = \frac{7}{9}$

$\frac{1}{9} + x = \frac{7}{9}$ since $xy = \frac{1}{9}$

$x = \frac{6}{9} = \frac{2}{3}$

From 3^{rd} equation:

$xy + y = \frac{5}{18}$

$\frac{1}{9} + x = \frac{7}{18}$

$y = \frac{5}{18} - \frac{2}{18} = \frac{3}{18} = \frac{1}{6}$

$\therefore (x+1)(y+1) = (\frac{2}{3}+1)(\frac{1}{6}+1) = \frac{5}{3} \cdot \frac{7}{6} = \frac{35}{18}$

8. Five positive integers are listed in increasing order. The difference between any two consecutive numbers in the list is three. The fifth number is a multiple of the first number. How many different such lists of five integers are there? (2007 Cayley #22)

Suppose the first number in the list is x.
Then the list would be: $x, x+3, x+6, x+9, x+12$.
$x + 12 = kx$, where k is an integer
$\therefore \frac{x+12}{x} = 1 + \frac{12}{x}$ is an integer
Therefore x is a positive divisor of 12. $\therefore x = 1, 2, 3, 4, 6,$ or 12
$\therefore$ There are 6 different lists.

9. Suppose that a, b and c are three numbers with

$$\begin{aligned} a+b&=3\\ ac+b&=18\\ bc+a&=6 \end{aligned}$$

Determine the value of c (2009 Cayley #22).

Equations 2 + 3:

$ac+b+bc+a=24$
$(a+b)c+(a+b)=24$

since $a+b=3$:

$3c+3=24$
$3c=21$
$c=7$

10. Solve the following system of equations (2000 Cayley #25 - modified)

$$\begin{aligned} x+xy+xy^2&=26\\ x^2y+x^2y^2+x^2y^3&=156 \end{aligned}$$

$x(1+y+y^2)=26 \qquad (1)$
$x^2y(1+y+y^2)=156 \qquad (2)$

$x\neq 0$ since $0\neq 26$
(2) $\div$ (1):

$$\frac{x^2y(1+y+y^2)}{x(1+y+y^2)} = \frac{156}{26}$$

$\therefore xy=6 \implies y=6/x$ and $x=6/y$

Substitute into (1):

$\frac{3}{y}(1+y+y^2)=13$

$3+3y+3y^2=13y$

$3y^2-10y+3=0$

$$y=\frac{10\pm\sqrt{10^2-4(3)(3)}}{2(3)}=\frac{10\pm 8}{6}=3\ ,\ 1/3$$

$y=3 \implies x=2$
$y=1/3 \implies x=18$
Therefore the solutions are $(x,y)=(2,3),(18,\frac{1}{3})$