Intermediate Math Circles
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Contest Prep: Harder Problems

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Today

What we will be doing

- continuing contest preparation
- working on harder problems

Prerequisites

- willingness to try
- persistence

Hidden themes

- problem solving strategies
- contest tips

Two Warm-up Problems

1. How many integers n are there where $1 \leq n \leq 100$ and n^n is a perfect square?

2. If $\dfrac{x^2+1}{x} = 1$, what is the value of $\dfrac{x^{99}+1}{x^{33}}$?

Warm-up Problem 1 Solution

Suppose n is even.
Then $n = 2k$ for some positive integer k.
In this case, $n^n = n^{2k} = (n^k)^2$, a perfect square.
There are 50 even integers n such that $1 \leq n \leq 100$.

Suppose n is an odd perfect square.
Then $n = m^2$ for some odd integer m.
In this case, $n^n = (m^2)^n = (m^n)^2$, a perfect square.
There are 5 odd perfect squares such that $1 \leq n \leq 100$.
(1, 9, 25, 49 and 81).

There are 55 integers in total with this property in this range.

Can you *prove* that the other 45 values of n do not give a perfect square?

Warm-up Problem 2 Solution

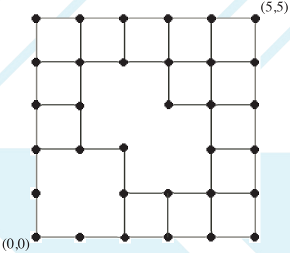
Note that $x^3 + 1 = (x+1)(x^2 - x + 1)$.
We have that $x^2 + 1 = x$ or $x^2 - x + 1 = 0$.

Substituting in the equation above, we obtain $x^3 + 1 = 0$ or $x^3 = -1$.

Thus, $\dfrac{x^{99}+1}{x^{33}} = \dfrac{(x^3)^{33}+1}{(x^3)^{11}} = \dfrac{(-1)^{33}+1}{(-1)^{11}} = \dfrac{-1+1}{-1} = 0$.

Ants

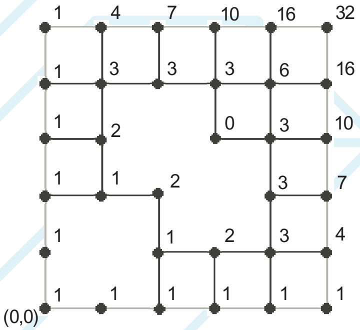
An ant is walking along the give piece of graph paper. She can travel only from a point shown to another point shown and can only change direction at these points. Someone has spilled water on the paper, erasing the points $(1, 1)$, $(2, 3)$ and $(3, 2)$. The ant begins at $(0, 0)$ and wants to travel to $(5, 5)$. How many different paths can the ant take if she can only travel right and up?



Ants Solution

In the diagram, each point is labelled with the number of possible paths to the point.

We determine each label by adding the labels immediately to the left and below. This is because when the ant reaches the point (x, y), she either went up or right to get there.



Therefore, the number of ways she can reach the point (x, y) is the number of ways of reaching $(x, y-1)$ plus the number of ways of reaching $(x-1, y)$. The final answer is 32.

Remainder

Give the remainder when

$$7^{700}2^{200} + 184000700005600042000280001400002018 - 14^{2017}$$

is divided by 14.

Remainder Solution

First notice that 14 divides the following integers.

$$7^{700}2^{200} = 7 \cdot 2 \cdot (7^{699}2^{199}) = 14 \cdot (7^{699}2^{199})$$

$$14 = 14 \cdot 1$$

$$28 = 14 \cdot 2$$

$$42 = 14 \cdot 3$$

$$56 = 14 \cdot 4$$

$$70 = 14 \cdot 5$$

$$84 = 14 \cdot 6$$

$$14^{2017} = 14 \cdot 14^{2016}$$

This means we need to find the remainder when $10^{34} + 2018$ is divided by 14.

Remainder Solution

Now, $2018 = 144(14) + 2$, and:

$$10^1 = 0(14) + 10$$

$$10^2 = 7(14) + 2$$

$$10^3 = 70(14) + 20 = 71(14) + 6$$

$$10^4 = 710(14) + 60 = 714(14) + 4$$

$$10^5 = 7140(14) + 40 = 7142(14) + 12$$

$$10^6 = 71420(14) + 120 = 71428(14) + 8$$

$$10^7 = 71428(14) + 80 = 71433(14) + 10$$

The pattern now repeats in a cycle of length 6, and $34 = 4 + 5(6)$ so the remainder when 10^{34} is divided by 14 is 4.
Therefore the desired final remainder is $2 + 4 = 6$.

Schedules

Six soccer teams are competing in a tournament in Waterloo. Every team is to play three games, each against a different team. (Note that not every pair of teams plays a game together.) Judene is in charge of pairing up the teams to create a schedule of games that will be played. Ignoring the order and times of the games, how many different schedules are possible?

Schedules Solution

Label the teams A, B, C, D, E, F.
Without loss of generality, there are two possible configurations:

1. A, B, C each play three games against each of D, E, F.
2. A, B, C each play each other, D, E, F each play each other, A plays D, B plays E and C plays F.

Note that there are $\left(\frac{1}{2}\right) \times \frac{6\times5\times4}{3\times2} = 10$ ways to choose A, B and C.
Therefore there are

- 10 ways to assign teams for the first configuration, and
- $10 \times 3 \times 2 = 60$ ways to assign teams for the second configuration.

This gives a total of 70 different schedules possible.