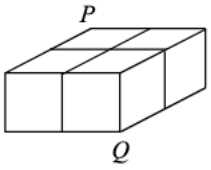
Intermediate Math Circles
February 19, 2020
Contest Prep: Harder Problems

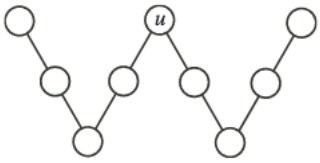
1. Grid lines are drawn on three faces of a rectangular prism, as shown. A squirrel walks from P to Q along the edges and grid lines in such a way that she is always getting closer to Q and farther away from P. How many different paths from P to Q can the squirrel take?



2. Nadia walks along a straight path that goes directly from her house (N) to her Grandmother's house (G). Some of this path is on flat ground, and some is downhill or uphill. Nadia walks on flat ground at 5 km/h, walks uphill at 4 km/h, and walks downhill at 6 km/h. It takes Nadia 1 hour and 36 minutes to walk from N to G and 1 hour and 39 minutes to walk from G to N. If 2.5 km of the path between N and G is on flat ground, what is the total distance from N to G?

3. How many triples (a, b, c) of positive integers satisfy the conditions $6ab = c^2$ and $a < b < c \leq 35$?

4. The smallest of nine consecutive integers is 2012. These nine integers are placed in the circles to the right. The sum of the three integers along each of the four lines is the same. If this sum is as small as possible, what is the value of u?



5. A square array of dots with 10 rows and 10 columns is given. Each dot is coloured either blue or red. Whenever two dots of the same colour are adjacent in the same row or column, they are joined by a line segment of the same colour as the dots. If they are adjacent but of different colours, they are then joined by a green line segment. In total, there are 52 red dots. There are 2 red dots at corners, with an additional 16 red dots on the edges of the array. The remainder of the red dots are inside the array. There are 98 green line segments. How many blue line segments are there?

6. Starting with the input (m, n), Machine A gives the output (n, m).
Starting with the input (m, n), Machine B gives the output $(m + 3n, n)$.
Starting with the input (m, n), Machine C gives the output$(m - 2n, n)$. Natalie starts with the pair $(0, 1)$ and inputs it into one of the machines. She takes the output and inputs it into any one of the machines. She continues to take the output that she receives and inputs it into any one of the machines. (For example, starting with $(0, 1)$, she could use machines B, B, A, C, B in that order to obtain the output $(7, 6)$.) Which of the following pairs is impossible for her to obtain after repeating this process any number of times?

(A) $(2009, 1016)$ **(B)** $(2009, 1004)$ **(C)** $(2009, 1002)$
(D) $(2009, 1008)$ **(E)** $(2009, 1032)$

7. Let p be the probability that, in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails. Determine the value of p.

8. Determine the sum of the number of digits in 2^{2016} plus the number of digits in 5^{2016}.

9. A teacher needs to place T identical tests on desks. There are D desks and they are arranged in a single row. No two tests can be placed on the same desk or on desks that are right beside each other. For which of the following values of T and D does the teacher have the largest number of ways to do this?

(A) $D = 10$ and $T = 3$ **(B)** $D = 11$ and $T = 4$ **(C)** $D = 12$ and $T = 5$
(D) $D = 13$ and $T = 2$ **(E)** $D = 17$ and $T = 8$

10. How many pairs (x, y) of integers are there such that $\dfrac{1}{x} + \dfrac{x}{y} + \dfrac{253}{xy} = 1$?