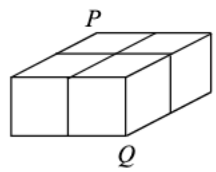
Intermediate Math Circles
February 19, 2020
Contest Prep: Harder Problems

1. Grid lines are drawn on three faces of a rectangular prism, as shown. A squirrel walks from P to Q along the edges and grid lines in such a way that she is always getting closer to Q and farther away from P. How many different paths from P to Q can the squirrel take?



There are 14 paths from P to Q that the squirrel can take. This can be solved using the same technique we used to solve the Ants problem.

This problem was Question 21 on the 2016 Pascal Contest and so a full solution can be found on the CEMC website.

2. Nadia walks along a straight path that goes directly from her house (N) to her Grandmother's house (G). Some of this path is on flat ground, and some is downhill or uphill. Nadia walks on flat ground at 5 km/h, walks uphill at 4 km/h, and walks downhill at 6 km/h. It takes Nadia 1 hour and 36 minutes to walk from N to G and 1 hour and 39 minutes to walk from G to N. If 2.5 km of the path between N and G is on flat ground, what is the total distance from N to G?

The total distance from N to G is 7.9km.

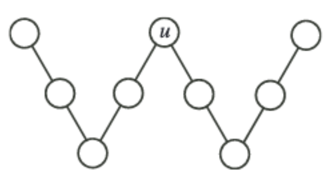
This problem was Question 24 on the 2014 Pascal Contest.

3. How many triples (a, b, c) of positive integers satisfy the conditions $6ab = c^2$ and $a < b < c \leq 35$?

There are 8 triples that satisfy these conditions.

This problem was Question 23 on the 2015 Pascal Contest.

4. The smallest of nine consecutive integers is 2012. These nine integers are placed in the circles to the right. The sum of the three integers along each of the four lines is the same. If this sum is as small as possible, what is the value of u?



The sum is as small as possible when $u = 2015$.

This problem was Question 24 on the 2012 Pascal Contest.

5. A square array of dots with 10 rows and 10 columns is given. Each dot is coloured either blue or red. Whenever two dots of the same colour are adjacent in the same row or column, they are joined by a line segment of the same colour as the dots. If they are adjacent but of different colours, they are then joined by a green line segment. In total, there are 52 red dots. There are 2 red dots at corners, with an additional 16 red dots on the edges of the array. The remainder of the red dots are inside the array. There are 98 green line segments. How many blue line segments are there?

   There are 37 blue line segments.

   This problem was Question 25 on the 2011 Fermat Contest.

6. Starting with the input (m, n), Machine A gives the output (n, m).
   Starting with the input (m, n), Machine B gives the output $(m + 3n, n)$.
   Starting with the input (m, n), Machine C gives the output$(m - 2n, n)$. Natalie starts with the pair $(0, 1)$ and inputs it into one of the machines. She takes the output and inputs it into any one of the machines. She continues to take the output that she receives and inputs it into any one of the machines. (For example, starting with $(0, 1)$, she could use machines B, B, A, C, B in that order to obtain the output $(7, 6)$.) Which of the following pairs is impossible for her to obtain after repeating this process any number of times?

   (A) $(2009, 1016)$ **(B)** $(2009, 1004)$ **(C)** $(2009, 1002)$
   (D) $(2009, 1008)$ **(E)** $(2009, 1032)$

   The correct answer is (D).

   This problem was Question 25 on the 2009 Pascal Contest.

7. Let p be the probability that, in the process of repeatedly flipping a fair coin, one will encounter a run of 5 heads before one encounters a run of 2 tails. Determine the value of p.

   We use T to represent tails and H to represent heads.
   We call a sequence of tosses successful if it ends with $HHHHH$ and does not contain TT.
   Let p_H be the probability that a given sequence is successful and begins with H and let p_T be the probability that a given sequence is successful and begins with T.
   Note that $p_T = \frac{1}{2}p_H$.
   Now a successful sequence beginning with H begins with HT, HHT, $HHHT$ or $HHHHT$ followed by a successful sequence, or it begins with $HHHHH$. Thus, $p_H = \frac{1}{4}p_H + \frac{1}{8}p_H + \frac{1}{16}p_H + \frac{1}{32}p_H + \frac{1}{32}$.
   Solving gives $p_H = \frac{1}{17}$.
   Therefore, $p_T = \frac{1}{32}$.
   The required probability is $p = p_T + p_H = \frac{1}{17} + \frac{1}{34} = \frac{3}{34}$.

8. Determine the sum of the number of digits in 2^{2016} plus the number of digits in 5^{2016}.

   Let r be the number of digits for 2^{2016} and s the number of digits for 5^{2016}. Then

   $$10^{r-1} < 2^{2016} < 10^r$$

   and

   $$10^{s-1} < 5^{2016} < 10^s$$

   so that

   $$(10^{r-1})(10^{s-1}) < (2^{2016})(5^{2016}) < (10^r)(10^s)$$

   or

   $$10^{r+s-2} < 10^{2016} < 10^{r+s}$$

   Thus, $r + s - 1 = 2016$ and so $r + s = 2017$.

9. A teacher needs to place T identical tests on desks. There are D desks and they are arranged in a single row. No two tests can be placed on the same desk or on desks that are right beside each other. For which of the following values of T and D does the teacher have the largest number of ways to do this?

(A) $D = 10$ and $T = 3$ **(B)** $D = 11$ and $T = 4$ **(C)** $D = 12$ and $T = 5$
(D) $D = 13$ and $T = 2$ **(E)** $D = 17$ and $T = 8$

Let $f(D, T)$ be the number of ways of arranging tests on desks.Then

$$f(D,T) = \begin{cases} D & T = 1 \\ 0 & 2T - 1 > D \\ f(D-2, T-1) + f(D-1, T) & \text{otherwise} \end{cases}$$

The first case is because when there is one test, it can be placed on any desk.
In the second case, there are not enough desks to place the tests so that no two are right beside each other.
The third case comes from considering when there is a test on the first desk and not the second (giving $f(D-2, T-1)$ ways in total) and alternatively when there is not a test on the first desk (giving $f(D-1, T)$ ways in total).
This recursive formula can be used to calculate the number of ways in each case by working up from small values of D and T.
In the table below, the entry in row D and column T gives the value of $f(D, T)$.

1	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0
4	3	0	0	0	0	0	0
5	6	1	0	0	0	0	0
6	10	4	0	0	0	0	0
7	15	10	1	0	0	0	0
8	21	20	5	0	0	0	0
9	28	35	15	1	0	0	0
10	36	56	35	6	0	0	0
11	45	84	70	21	1	0	0
12	55	120	126	56	7	0	0
13	66	165	210	126	28	1	0
14	78	220	330	252	84	8	0
15	91	286	495	462	210	36	1
16	105	364	715	792	462	120	9
17	120	455	1001	1287	924	330	45

(Given that this is multiple choice, we can also eliminate most cases pretty quickly.)

10. How many pairs (x, y) of integers are there such that $\frac{1}{x} + \frac{x}{y} + \frac{253}{xy} = 1$?

The equation can be rearranged to get $(x-1)(y-x-1) = 254 = -2 \times 127$.
This prime factorization of 254 tells us that it has 4 positive divisors and 8 divisors in total.
We know that $x - 1$ must be a divisor of 254 so we can check all 8 possibilities.
Doing this reveals that that there are 7 ordered pairs of integers that satisfy the given equation because we have to rule out possibilities where $x = 0$ or $y = 0$.