Here are some mathematical games. In each case, try to figure out a winning strategy. If you finish early, think about the extra questions for each game. Have fun!

1. **Thai 21**

   There are 21 flags. Players take turns removing 1, 2, or 3 flags. The player who takes the last flag wins.

   - Do you want to go first or second?
   - What if the person who takes the last flag loses?
   - What if there are only 20 flags?

   Questions:

   What is an *impartial combinatorial game*?

   What is a *winning strategy*?

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2. **Pick and Split**

   There are two piles of stones. One has 10 stones, and the other has 13 stones. On each players turn, they must discard one pile of stones and split the other pile into two piles. Every pile must have at least one stone. The first player unable to make a legal move loses. In other words, the game ends when there are two piles of one stone each.
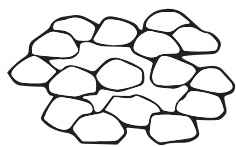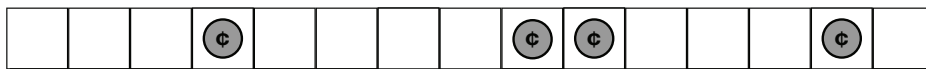
   - Do you want to go first or second?
   - What if the piles have 11 stones and 13 stones?
   - What if the piles have n stones and m stones?
   - What if there are 3 piles of stones?

3. **Sliding Pennies**

The game begins with four pennies placed on a 1×15 grid, just like the picture below.

Players take turns moving one penny to the right, but may not slide pennies past each other. For instance, a legal move for Player 1 might be the following.


The first player unable to make a move loses. In other words, the game ends when the pennies are stacked all the way on the right side of the grid.

- Do you want to go first or second?
- What if the pennies are arranged in a different starting position? Can you tell whether you want to go first or second?

4. **Half a Chessboard**

There is a 4×8 (vertical) chessboard with four pawns in the top row. On each turn, players can move $1, 2, 3$, or all 4 pawns down one square, until they reach the bottom row. The player who cannot make a move loses.
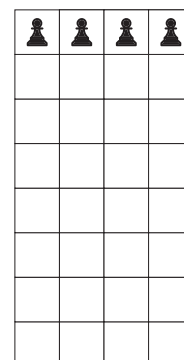
- Do you want to go first or second?

- What if the pawns are allowed to start in any position (one in each column)?

- What if there are 5 pawns on a 5×8 chessboard? What if there are 4 pawns on an $n \times 8$ chessboard?

Questions:

What are some *similarities* between these four games? What are some *differences*?

More Problems!

If you liked those games, see if you can figure out winning strategies for these ones. Some of them are tricky!

5. **Erase from 13**

   A chalkboard has the numbers 1,2,3,...,13 written on it. Two players take turns erasing a number from the board, until two numbers remain. The first player wins if the sum of the last two numbers is a multiple of 3. Otherwise, the second player wins.

   What if we start with the numbers $1, 2, 3, ..., 2020$?

6. **Generalized Thai 21**

   There are n flags. On their turn, a player can remove 1, 2, 3,..., or k flags. The player who takes the last flag wins.

7. **Even More Thai 21**

   There are 21 flags. On their turn, a player can remove 1, 2, or 4 flags. The player who takes the last flag wins.

   There are n flags. On their turn, a player can remove 1, 3, or 5 flags. The player who takes the last flag wins.

8. **Sliding Nickels**

   Just like **Sliding Pennies**, the games begins with 5 (not 4!) nickels placed on a 1×15 grid. Players take turns sliding one nickel to the right, but may not slide nickels past each other. The first player unable to make a legal move loses.

9. **The Subtracting Game**

   Similar to **Thai 21**, this game starts with 44 flags. Player 1 can remove any number of flags, but must leave at least one. Thereafter, players may remove *at most* as many as the previous player did. The player who takes the last flag wins.

10. **Yet Another Flag Game**

    Like **The Subtracting Game**, this game starts with 44 flags. Player 1 can remove any number of flags, but must leave at least one. Thereafter, players may remove *up to twice as many* as the previous player did. The player who takes the last flag wins.