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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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Black Holes

Scientific Notation

Let's look at some exponents:

$10^2 = 10 \times 10$
$10^3 = 10 \times 10 \times 10$
$10^5 = 10 \times 10 \times 10 \times 10 \times 10$

What about 10^{-2}?

Rule: $10^{-n} = \frac{1}{10^n}$
So, $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$

What about $10^2 \times 10^3$?

$= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$
We got 2 tens from 10^2 and 3 tens from 10^3.

What about $10^{-2} \times 10^3$?

$\frac{\cancel{10} \times \cancel{10} \times 10}{\cancel{10} \times \cancel{10}} = 10^1$

What do you notice?

Our answers always had the same base and the exponents of the answers were the sums of the two exponents being multiplied.
ie. $10^m \times 10^n = 10^{m+n}$

What about $(10^3)^2$?

$(10 \times 10 \times 10) \times (10 \times 10 \times 10) = 10^6$

What do you notice?

Our exponent in our answer is the product of the two exponents.
ie. $(10^m)^n = 10^{m \times n}$

What about $10^5 \div 10^3$?

$\frac{\cancel{10} \times \cancel{10} \times \cancel{10} \times 10 \times 10}{\cancel{10} \times \cancel{10} \times \cancel{10}} = 10^2$

What about $10^5 \div 10^7$?

$\frac{\cancel{10}\times\cancel{10}\times\cancel{10}\times\cancel{10}\times\cancel{10}}{\cancel{10}\times\cancel{10}\times\cancel{10}\times\cancel{10}\times\cancel{10}\times 10\times 10} = \frac{1}{10^2} = 10^{-2}$

What do you notice?

The exponent in the answer is the exponent of the dividend minus the exponent of the divisor.
ie. $\frac{10^m}{10^n} = 10^m \div 10^n = 10^{m-n}$

Now let's try something else. What if we have 1.1×10? What is this equal to?

Using a calculator, or manually adding $1.1 + 1.1 + 1.1....$ ten times, we get that our answer is 11.

So, we see that multiplying by 10^1 moves the decimal place over ONCE to the RIGHT.

$1.1 \rightarrow 11$

What if we have 0.5×10^2?

$\frac{1}{2} \times 100 = 50$

So, we see that multiplying by 10^2 moves the decimal place over TWICE to the RIGHT.

$0.50 \rightarrow 50$

What if we have 5×10^{-1}?

$5 \times \frac{1}{10} = \frac{1}{2} = 0.5$

So, we see that multiplying by 10^{-1} moves the decimal place over ONCE to the LEFT.

$5 \rightarrow 0.5$

What if we have 5×10^{-2}?

$5 \times \frac{1}{10^2} = \frac{5}{100} = 0.05$

So, we see that multiplying by 10^{-2} moves the decimal place over TWICE to the LEFT.

$5 \rightarrow 0.05$

What pattern do you notice?

The exponent tells us how many times to move the decimal place over. If the exponent is positive, it tells us to move the decimal to the right. If the exponent is negative, it tells us to move the decimal to the left.

Let's apply these ideas:

Multiply $(5.5 \times 10^3) \times (6.1 \times 10^4)$.

Collect like terms.
$(5.5 \times 6.1) \times (10^3 \times 10^4)$. Multiply the things in the brackets seperately.
$= 33.55 \times 10^7$
$= 3.355 \times 10^8$. (Scientific notation requires we only have one digit before the decimal. Because we had to move it once to the left, we compensate for this change by adding one to the exponent, telling us to move back one to the right).

What if we have $(3.0 \times 10^8)^2$?

Let's square both parts seperately.
$(3.0)^2 \times (10^8)^2$
$= 9.0 \times 10^{16}$

Exercises:

1. $(4.4 \times 10^{-4}) \times (8.6 \times 10^{9})$

   $(4.4 \times 8.6) \times (10^{-4} \times 10^{9})$
   $=37.84 \times 10^{5}$
   $=3.784 \times 10^{6}$

2. $(2.5 \times 10^{1}) \div (1.4 \times 10^{3})$

   $(2.5 \div 1.4) \times (10^{1} \div 10^{3})$
   $= 1.79 \times 10^{-2}$

3. $(6.67 \times 10^{-11}) \div (3.0 \times 10^{8})^{2}$

   $(6.67 \times 10^{-11}) \div (9.0 \times 10^{16})$
   $= (6.67 \div 9.0) \times (10^{-11} \div 10^{16})$
   $= 0.74 \times 10^{-27}$
   $= 7.4 \times 10^{-28}$ (Scientific notation requires that we have 1 non-zero digit before the decimal. So, we moved the decimal place over once to the right and to compensate for this change we subtracted one from the exponent).

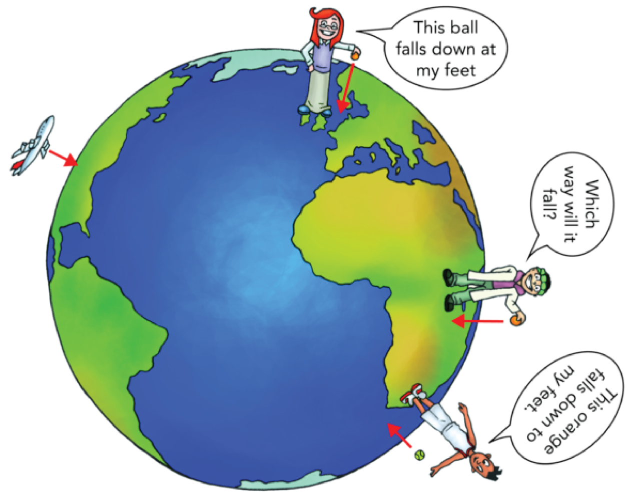
Recap:

$10^{m} \times 10^{n} =$	10^{m+n}
$\frac{10^{m}}{10^{n}} =$	10^{m-n}
$(10^{m})^{n} =$	$10^{m \times n}$
5.0×10^{m}, where m is positive.	Decimal moves to the right m place values.
5.0×10^{-m}, where m is positive.	Decimal moves to the left m place values.

Gravity

What is gravity?

When we talk about the Earth's gravity, we describe it as a ____force of attraction____ where the Earth pulls everything towards its ____centre____. This is how we are able to stay on the ground rather than float, and it is also why when we throw a ball into the air it does not just keep going up forever, but rather it falls back down toward the ground.



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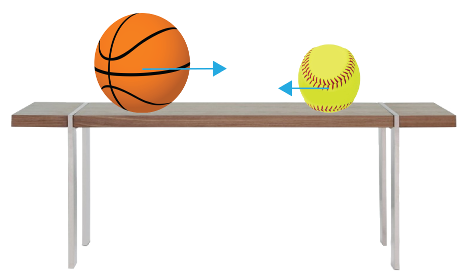
Is the Earth the only object that has gravity?

____No!! All objects have gravity!!____

That is, ____every____ object has a force of attraction that pulls other objects towards its centre.



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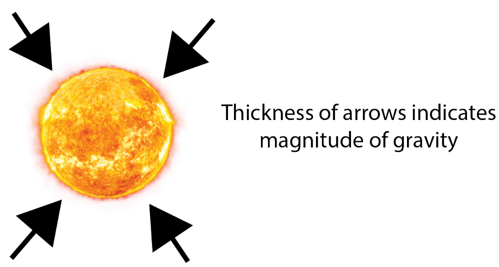
Can you list some examples of objects that have gravity?

...But wait, if all objects have gravity, why can't I see my water bottle being pulled towards my laptop when I put them both on a table close to each other? Shouldn't the water bottle be pulling my laptop towards it? Or vice versa?

The answer to this is that the water bottle actually ___IS___ attracting my laptop to it due to its ___gravity___, and vice versa my laptop is also attracting my water bottle towards it due to its gravity! Why can't we see this?



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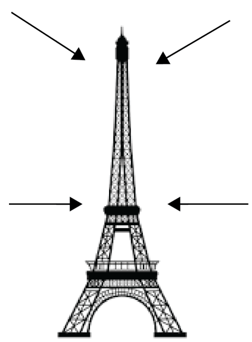
The more ___massive___ an object is, the ___greater___ its gravity will be. That is, objects with mass as great as the Earth, sun etc. have very strong forces of gravity, so we can see the effect of the Earth's gravity on a baseball when it falls to the ground easily, as the Earth's gravity is so much stronger than the baseball's.



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Objects that have much ___less___ mass than the Earth, such as my laptop, water bottle or even the hugest buildings, do ___not___ have enough mass for objects to actually be ___pulled___ towards them. Although, ___every object___ still ___attracts___ all other objects to it at least a little, just usually not enough for objects to actually move towards it.

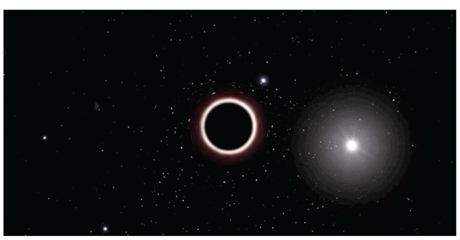
Imagine this:

- The mass of the Earth is ____5.972×10^{24} kg____. (That's 5972000000000000000000000 kg.)
- The mass of the sun is ____1.99×10^{30} kg____. That's 333,000 times heavier than the Earth.
- Some stars however, have masses that are much greater than the mass of our sun. If a star has a mass of at least ____10____ times the mass of our sun, when the star dies (burns out), all of its mass gets compressed into such a small space that it creates a __black hole__.

What's a black hole?

A black hole is an object in outer space whose gravity is so strong that once another object gets close enough to it, it is trapped inside the black hole forever and can never escape. This is because black holes have so much mass that they pull everything towards it with a huge amount of force.

What does a black hole look like?



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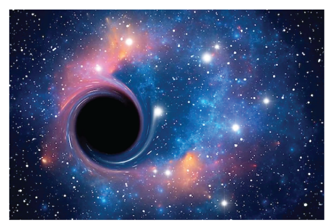
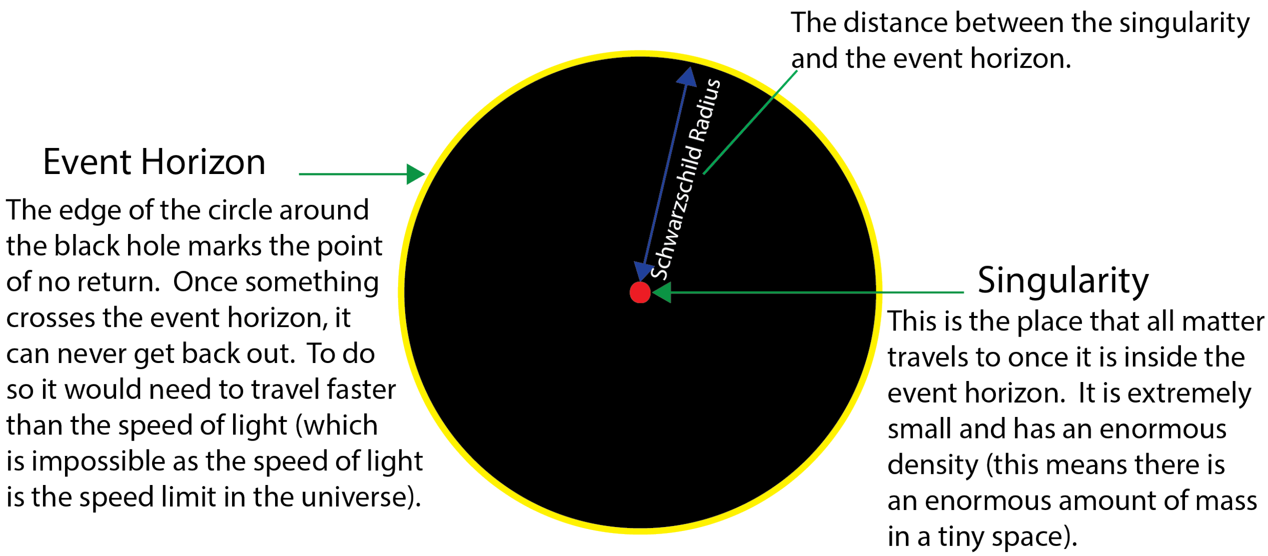


Retrieved from blogs.scientificamerican.com

Diagram of a black hole:



Quick Review:

The radius of a circle is the distance between the centre of the circle and the edge of the circle.

Likewise, the radius of an sphere is the distance from the centre of the sphere to the surface (edge) of the sphere.

What is so special about the Schwarzschild radius?

Anything can become a black hole. That's right, anything!

Write down a few things that we could theoretically compress into a black hole:

The catch here is that, in order to make something into a black hole, we need to compress (squeeze) it into a really small space. The Schwarzchild radius tells us how small we would need to compress an object to turn it into a black hole, depending on its mass.

The Schwarzschild radius equation:

$$R = \frac{2GM}{c^2}$$

Retrieved from https://www.perthobservatory.com.au/news/the-schwarzschild-radius

Where,
R = The Schwarzschild radius in m (the radius that we must squish the object to).
$G = 6.67 \times 10^{-11}\text{Nm}^2/\text{kg}^2$(gravitational constant).
M = The mass of the object in kg.
$c = 3 \times 10^8$ m/s (speed of light).

Example:

The mass of our moon is 7.35×10^{22} kg. What is its Schwarzschild radius?

$R = \frac{2\times 6.67\times 10^{-11}\times 7.35\times 10^{22}}{(3.0\times 10^8)^2}$
$R = \frac{2\times 6.67\times 7.35}{9} \times \frac{10^{-11}\times 10^{22}}{10^{16}}$
$R = 1.09 \times 10^{-4}$ m.

Exercise:

The mass of the Earth is 5.98×10^{24} kg. What radius must you squish the Earth into in order to turn it into a black hole?

$R = \frac{2 \times 6.67 \times 10^{-11} \times 5.98 \times 10^{24}}{(3.0 \times 10^8)^2}$
$R = \frac{2 \times 6.67 \times 5.98}{9} \times \frac{10^{-11} \times 10^{24}}{10^{16}}$
$R = 8.86 \times 10^{-3}$ m.

An Alternate Schwarzschild Equation:

Because the value of G and c in the Schwarzschild equation are always the same, we can simplify the equation to:

$$R = 1.48 \times 10^{-27} M$$

Where,
R = The Schwarzschild radius in m (the radius that we must squish the object to).
M = The mass of the object in kg.
1.48×10^{-27} is just $\frac{2G}{c^2}$.

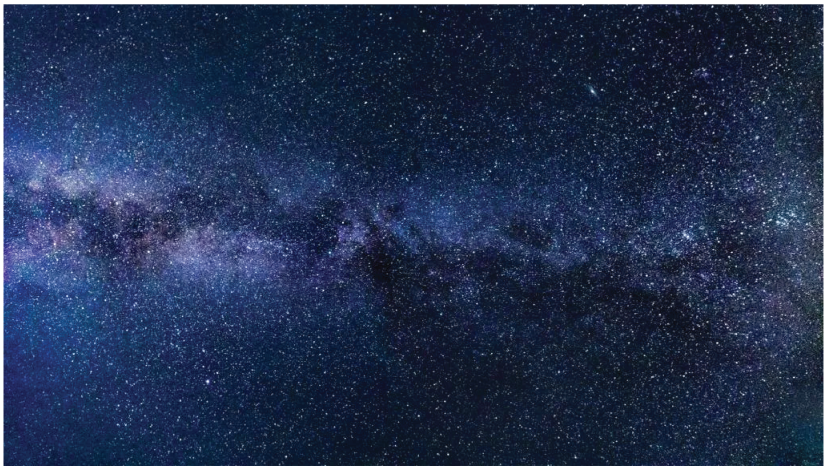
Try using this simplified formula to calculate the Schwarzschild radius of the moon again (7.35×10^{22} kg). Do you get the same answer?

$R = 1.48 \times 10^{-27} \times 7.35 \times 10^{22}$
$R = 1.48 \times 7.35 \times 10^{-27} \times 10^{22}$
$R = 1.09 \times 10^{-4}$ m.

A Closer Look at The Formation of Black Holes

We mentioned briefly that black holes form when a star dies, if that star had a great enough mass.

What does it mean for a star to die?



Retrieved from https://www.lightstalking.com/night-sky-photography/

When we look up at the night sky, we see stars as specs of light. This is essentially what a star is, a ball of light in the sky. But a star needs to possess certain chemicals that react in order to emit this light. When a star runs out of these chemicals, it explodes and transforms into either a different type of star or a black hole, so we say that it "dies."

A quick rundown:

- We have three types of black holes, classified by their size: stellar-mass black holes, intermediate-mass black holes, and supermassive black holes.
- Stars that are less than 10 (or 1.0×10^1) times as massive as our sun, will not form a black hole when they die.
- Stars that are between 10 (or 1.0×10^1) and 100 (or 1.0×10^2) times as massive as our sun will form a stellar-mass black hole when they die.
- Stars cannot form black holes bigger than this, however there are other ways that even bigger black holes can be created.

- We call a black hole that is between ___100 (or 1.0×10^2)___ and ___1,000,000 (or 1.0×10^6)___ times as massive as our sun an ___intermediate-mass___ black hole.
- We call a black hole that is greater than ___1,000,000 (or 1.0×10^6)___ times the mass of our sun a ___supermassive___ black hole.

Example:

The mass of our sun is 1.99×10^{30} kg. Will a star with a mass of 2.189×10^{31} kg turn into a black hole when it dies?

We have to figure out how many times heavier this star is than the sun. To do this, we will use the equation,

$X = \frac{\text{Mass of the star in question}}{\text{Mass of the sun}}$

Where X is the amount of times more massive the star is than the sun.

So,

$X = \frac{2.189 \times 10^{31} \text{kg}}{1.99 \times 10^{30} \text{kg}}$

$X = 1.1 \times 10^{31-30}$

$X = 1.1 \times 10^1$

$X = 11$

Since a star has to be 10 or more times more massive than the sun to become a stellar - mass black hole when it dies, this particular star will become a stellar - mass black hole when it dies.

Example:

The mass of a black hole is found to be 1.592×10^{32} kg. Is this black hole massive enough to be considered an intermediate-mass black hole?

Let's use the equation, $Y = \frac{\text{Mass of the black hole in question}}{\text{Mass of the sun}}$

Where Y is the amount of times more massive the black hole is than the sun.

$Y = \frac{1.592 \times 10^{32} \text{kg}}{1.99 \times 10^{30} \text{kg}}$

$Y = 0.8 \times 10^{32-30}$

$Y = 0.8 \times 10^2$

$Y = 8.0 \times 10^1$

$Y = 80$

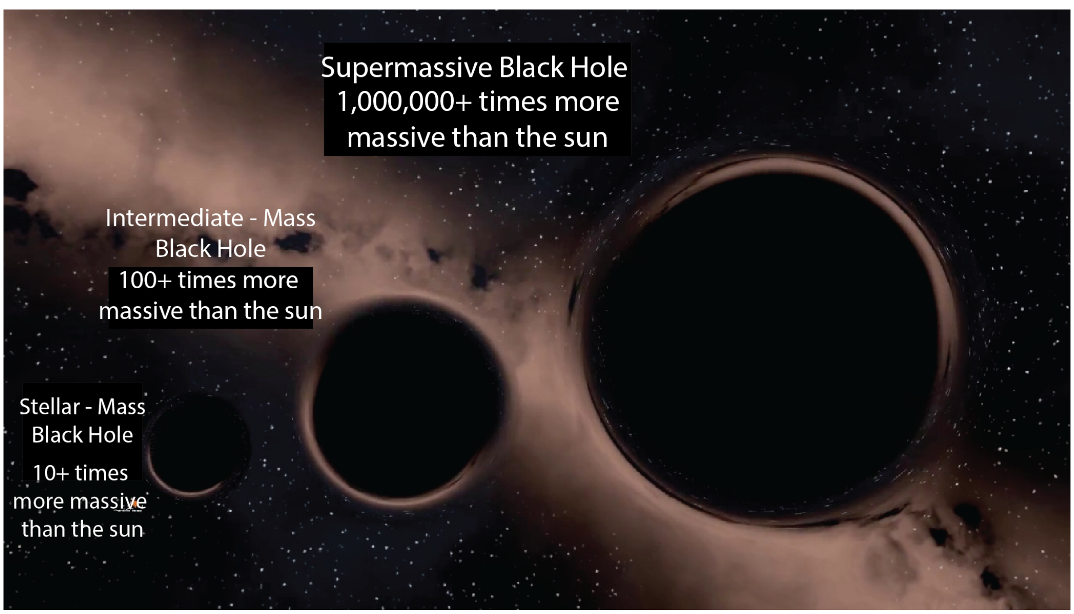
Since $Y = 80$, we know that this black hole is 80 times more massive than the sun. But it needs to be 100 (or 1.0×10^2) or more times more massive than our sun to be considered an intermediate - mass black hole. Therefore, this black hole would be a stellar - mass black hole, not an intermediate - mass black hole.

Example:

While searching through our Milky Way galaxy, you detect a black hole with a mass of 7.96×10^{36} kg. That's a really massive black hole, is it a supermassive black hole?

$Y = \frac{7.96 \times 10^{36} \text{kg}}{1.99 \times 10^{30} \text{kg}}$
$Y = 4 \times 10^{36-30}$
$Y = 4 \times 10^{6}$
$Y = 4{,}000{,}000$
Since Y is greater than 1,000,000 (or 1.0×10^{6}), we can say that it is indeed a supermassive black hole. This is actually approximately the mass of the supermassive black hole in the middle of our galaxy.

Recap!



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Exercises:

Classify each black hole as either a stellar-mass, intermediate or supermassive black hole.

1. A black hole with a mass of 9.95×10^{31} kg.

   $Y = \frac{9.95\times10^{31}\text{kg}}{1.99\times10^{30}\text{kg}}$
   $Y = 5 \times 10^{31-30}$
   $Y = 5 \times 10^{1}$
   $Y = 50$
   This black hole is between 10 and 100 (between 1.0×10^{1} and 1.0×10^{2}) times the mass of the sun, hence it is a stellar-mass black hole.

2. A black hole with a mass of 3.98×10^{36} kg.

   $Y = \frac{3.98\times10^{36}\text{kg}}{1.99\times10^{30}\text{kg}}$
   $Y = 2 \times 10^{36-30}$
   $Y = 2 \times 10^{6}$
   $Y = 2{,}000{,}000$
   This black hole is more than 1,000,000 (or 1.0×10^{6}) times the mass of the sun, hence it is a supermassive black hole.

3. A black hole with a mass of 1.17×10^{33} kg.

   $Y = \frac{1.17\times10^{33}\text{kg}}{1.99\times10^{30}\text{kg}}$
   $Y = 0.59 \times 10^{33-30}$
   $Y = 0.59 \times 10^{3}$
   $Y = 5.9 \times 10^{2}$
   $Y = 590$
   This black hole is between 100 and 1,000,000 (between 1.0×10^{2} and 1.0×10^{6}) times the mass of the sun, hence it is an intermediate-mass black hole.

Problem Set

* Indicates challenge problems

1. Use our scientific notation exponent rules to simplify the following:
   Ex. We would write $10^3 \times 10^6$ as 10^9.

   (A) $10^5 \times 10^9$ (B) $10^7 \times 10^{55}$ (C) $10^{-8} \times 10^{-10}$ (D) $10^5 \times 10^{-3}$

   (E) $\frac{10^{77}}{10^{60}}$ (F) $\frac{10^{43}}{10^{80}}$ (G) $\frac{10^{-22}}{10^5}$ (H) $\frac{10^{-25}}{10^{-3}}$

   (I) $(10^4)^2$ (J) $(10^{12})^5$ (K) $(10^3)^{-4}$

   (A) 10^{14} (B) 10^{62} (C) 10^{-18} (D) 10^2

   (E) 10^{17} (F) 10^{-37} (G) 10^{-27} (H) 10^{-22}

   (I) 10^8 (J) 10^{60} (K) 10^{-12}

2. Write the following as a single number:
   Ex. We would write 1.0×10^4 as 10,000.

   (A) 3.8×10^2 (B) 5.6×10^6 (C) 8.9×10^1 (D) 7.5×10^{-3}

   (E) 6.2×10^{-1}

   (A) 380 (B) 5,600,000 (C) 89 (D) 0.0075

   (E) 0.62

3. Put the following in scientific notation:

(A) 45,000 (B) 6,899,000 (C) 0.57 (D) 0.00032

(E) 50 (F) 468.9

(A) 4.5×10^4 (B) 6.899×10^6 (C) 5.7×10^{-1} (D) 3.2×10^{-4}

(E) 5.0×10^1 (F) 4.689×10^2

4. Calculate the following and express your answers in scientific notation.

(A) $5.8 \times 10^{-27} \times 5.5 \times 10^{30}$ (B) $6.4 \times 10^{44} \times 1.5 \times 10^{-8}$ (C) $\frac{5.7 \times 10^{19}}{4.2 \times 10^6}$

(D) $\frac{4.4 \times 10^9}{3.9 \times 10^{-5}}$ (E) $\frac{9.2 \times 10^{10} \times 7.6 \times 10^{-3}}{8.1 \times 10^4}$ (F) $\frac{1.6 \times 10^{-10} \times 5.6 \times 10^{-3}}{3.1 \times 10^{-6}}$

(G) $(5.4 \times 10^6)^2$ (H) $\frac{4.5 \times 10^4}{(6.3 \times 10^3)^2}$

(A) 3.2×10^4 (B) 9.6×10^{36} (C) 1.4×10^{13} (D) 1.1×10^{14}

(E) 8.6×10^3 (F) 2.9×10^{-7} (G) 2.9×10^{13} (H) 1.1×10^{-3}

5. Give an example of a scenario where we could see the effects of Earth's gravity.

Numerous possible answers. Some examples could be:
Throwing a ball into the air and watching it fall back down to the Earth, jumping and coming back down, being able to walk on the ground rather than float, apples falling off of trees onto the ground, it being easier to set things onto the ground than to lift them up.

6. Why do we not see the effects of a pencil's gravity?

   Because a pencil does not have a great enough mass for it to actually pull anything towards it, although it still attracts objects.

7. Which has a greater force of gravity: the Earth or the sun? Why?

   The sun because it has a greater mass. The more massive an object is, the greater its force of gravity.

8. Redraw the diagram of a black hole that we looked at in this lesson and explain to a friend or family member what each of the parts of a black hole are that we labelled.

9. You tell your friend that you can turn a piece of paper into a black hole. Your friend doesn't believe you, so you start crushing the paper up into a smaller and smaller space, trying to create a black hole. If your sheet of paper has a mass of 4.5×10^{-3} kg, what radius would you have to condense the paper to in order to create a black hole?

   6.67×10^{-30} m.

10. Pluto is upset about being deemed "not a planet" anymore. To get its revenge, Pluto decides to try to become a black hole so that it can swallow up the rest of the planets. Pluto has a mass of 1.31×10^{22} kg. To what radius must Pluto condense to in order to become a black hole?

    1.9×10^{-5} m.

11. *The Schwardzchild radius of my dog is 4.74×10^{-26} m. What is the mass of my dog in kg?

    Answer: 3.2×10^{1} or 32 kg.

    We have to rearrange the Schwardzchild radius equaiton for mass.

    We have $R = \frac{2GM}{c^2}$. Let's multiply both sides by c^2.

$Rc^2 = \frac{2GM}{c^2} \times c^2$. This cancels out c^2 on the right side of the equation. ($\frac{c^2}{c^2} = 1$).

Now we have $Rc^2 = 2GM$. Let's divide both sides by $2G$.

$\frac{Rc^2}{2G} = \frac{2GM}{2G}$. This cancels out $2G$ on the right side of the equation. ($\frac{2G}{2G} = 1$).

Now we have $\frac{Rc^2}{2G} = M$.

Sub in our values into this equation to get $M = 3.2 \times 10^1$ or 32 kg.

12. *What is the effect of decreasing the mass of the object on the Schwardzchild radius?

    Answer: It will cause the Schwardzchild radius to decrease as well.
    Looking at the equation $R = \frac{2GM}{c^2}$, we see that if we decrease M, this will decrease the value of the number on the right side of the equation. Since whatever we do to one side of the equation, we also do to the other, R will also decrease.

13. The following is a list of masses of different stars. Which of these stars will turn into a black hole when they die? (Remember that the mass of the sun is 1.99×10^{30} kg).

    (A) 1.22×10^{31} kg (B) 1.99×10^{31} kg (C) 8.32×10^{31} kg (D) 1.12×10^{31}kg

    (E) 1.32×10^{32} g

    Answer: B and C only.

14. The following is a list of masses of black holes. Classify which type of black hole each one is.

    (A) 2.5×10^{37} kg (B) 1.7×10^{36} kg (C) 6.5×10^{33} kg (D) 7.8×10^{34} kg

    (E) 9.0×10^{34} g

    (A) Supermassive (B) Intermediate - mass (C) Intermediate - mass

    (D) Intermediate - mass (E) Stellar - mass (Did you check your units?)

15. *You find a black hole that is 55 times as massive as the sun. What is the mass of this black hole?

Answer: 1.09×10^{32} kg.

We have the equation,

$\text{Y} = \frac{\text{Mass of the black hole in question}}{\text{Mass of the sun}}$

We can rearrange this equation for the mass of the black hole by multiplying both sides of the equation by the mass of the sun.

$\text{Y}\times\text{Mass of the sun} = \frac{\text{Mass of the black hole in question}}{\text{Mass of the sun}} \times \text{Mass of the sun}$

Now the mass of the sun cancels out on the right side of the equation and we get,

$\text{Y}\times\text{Mass of the sun} = \text{Mass of the black hole in question}$

Subbing in $\text{Y} = 55$, we calculate the mass of the black hole to be 1.09×10^{32} kg.

16. *Because the Earth has a strong force of gravity, in order to leave the Earth, rocketships have to be moving very fast. This speed that a rocketship must move to leave the Earth is called the *escape velocity*. Because the rocketship's speed has to counteract the Earth's force of gravity, we can imagine that the stronger the force of gravity of an object, the faster you would need to move to counteract that object's gravity and escape from the object. So, for example, the sun has a greater escape velocity than the Earth (Meaning that rocketships would need to move faster to leave the sun than they do to leave the Earth).

    The formula to find the escape velocity to leave any particular object is $v = \sqrt{\frac{2GM}{R}}$.

    Where,
    v is the escape velocity.
    G is the gravitational constant ($6.67 \times 10^{-11}\text{Nm}^2/\text{kg}^2$).
    M is the mass of the object that you (or another object) are trying to escape from.
    R is the radius of the object that you (or another object) are trying to escape from.

    Use this equation and the Schwardzchild equation to show that light cannot escape from a black hole.

    We can rearrange the Schwardzchild equation to get $c = \sqrt{\frac{2GM}{R}}$.
    Notice that this is the same as the escape velocity equation, where we know that M and R stand for the mass and radius of the black hole. The difference here is that for our speed, v, we have c, which we know is the speed of light. Therefore, this equation shows that the escape velocity of a black hole is the speed at which light moves. This implies that in order to escape a black hole, one must travel faster than the speed of light. Hence, light is not moving fast enough to escape a black hole.