UNIVERSITY OF
WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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To Infinity, and Beyond!

What is Infinity?

Retrieved from: www.edgemagazine.net

Infinity represents a quantity with no bounds. In math, we write infinity as ____________ and it represents something that is _____________________________________. However, infinity is not an actual number, but rather a concept. That is to say, it does not behave the same as a number would.

In our warm-up, we thought of numbers that got successively bigger. This list of numbers can go on ____________ and there is no one value that we can say is the "biggest." So, we say that an ____________ amount of numbers exist.

Exploring Infinity as a Concept

Buzz Lightyear is on a mission to travel to the edge of the universe. He sets off flying at the speed of light. He travels past Saturn, Neptune and out of our solar system. He passes out of our Milky Way galaxy and buzzes past millions more. When will he get to the edge of the universe?

Retrieved from: www.pinterest.ca

Well, we don't know for sure. However, many physicists think that the answer to this question is ____________ as many believe that the universe is ____________ in size. That is to say, the universe may go on ____________ in size. If this is true, Buzz Lightyear will ____________ reach the edge of the universe, because ____.

In order to imagine the size of the universe, you must allow yourself to imagine the largest thing you can possibly think of, imagine the universe taking up as much space as you can possibly think of - the universe is still even bigger than that.

Retrieved from: flamemedia.tv

Try to answer the following questions:

1. (a) An infinite number of people stand in line at Walmart. If one more person joins the line, how many people are now in line?
   (b) If 200 more people join the end of the line, how many people are now in line?
   (c) If an infinite number of people are also standing in line at Sobeys but Sobeys closes and everyone in this line joins the end of the Walmart line with an infinite amount of people in it, how many people are now in this line?
2. You have a lot of spare time one weekend and decide to bake an infinite number of cookies. If your friend steals 3 cookies from you, how many cookies do you have now?
3. (a) If I invite an infinite number of guests to my birthday party and each guest brings two of their friends, how many people are at my birthday party?
   (b) If I invite an infinite number of guests to my birthday party and each guest brings an infinite number of their friends, how many people are at my birthday party?

None of this makes sense? Let's try demonstrating some of these examples.

Hilbert's Hotel

Video

Retrieved from: intuitivescienceblog.wordpress.com

The Hilbert Hotel has an infinite amount of rooms. You and your friend go to check in to the Hilbert Hotel, but you are told that all of the rooms are full. Your friend starts to turn away, but you realize something and stop them. If the hotel has an infinite number of rooms, you can ask the people in each room to move to the next room. Everyone in room 1 would move to room 2, the people in room 2 would move to room 3, and so on forever. This would leave room number 1 free for you and your friend.

What would happen if an *infinite* amount of people wanted to check into the Hilbert Hotel, but the hotel was full?

Cardinality

Cardinality: The number of ____________ in a ____________.

Examples:

1. If I have a basket (or set) of 12 apples, what is the cardinality of this set?
2. Determine the cardinality of the following set of numbers: $\{1, 3, 5, 7, 9\}$.

Bijections

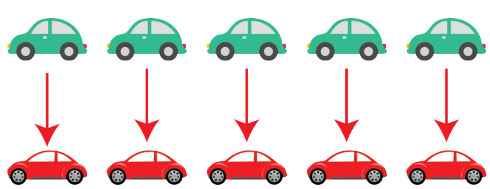
Bijection: There is a bijection between two sets when we can line up the two sets and ____________ every element of ____________ with exactly ____________ element of the other. This means that every element in each set is paired with exactly one element in the other, with no one element in either set left without a partner. If we can find a bijection between two sets, we say that two sets have the ________________________ (the same number of elements).

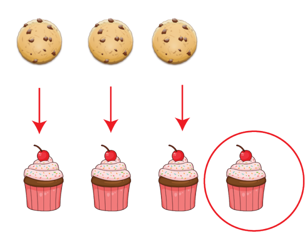
Example of a Bijection:

Show that the following two sets have the same cardinality: $\{1, 3, 5, 7, 9\}$, $\{2, 4, 6, 8, 10\}$

Exercise:

Which of the following are bijections:

1. 

2. 

3. I have 9 apples in my bag and I want to prove that the same number of positive integers exist as the number of apples that I have. How do I do this?

Infinite Sets

Infinite Set: Any set that has an ____________ number of ____________.

Examples of infinite sets include...

- The positive integers $\{1, 2, 3, 4, ...\}$
- All integers $\{... - 2, -1, 0, 1, 2, ...\}$
- Even numbers
- Odd numbers
- Square numbers $\{1, 4, 9, 16, 25, ...\}$

How can we compare the sizes of infinite sets?
Just like with our finite sets, we can use ____________ to show that two ____________ have the ____________ number of elements.

Galileo's Paradox

Galileo said that there are just as many ________________________ numbers as there are ________________________.

Galileo used a bijection to prove his claim:

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 4 & 9 & 16 & 25 & 36 & 49 & 64 & 81 & 100\ldots \end{array}$$

Here, we see that we are able to link each positive integer with a square number (1 and 1, 2 and 4, 3 and 9, etc.). We can just square each positive integer and get the corresponding perfect square number. Also, each perfect square links to one positive integer, we can square root the perfect square to get the positive integer. We can see that this will go on forever and we will always be able to pair each element in the first infinite set with exactly one element of the second, with no element left over in either set. Because we were able to create a bijection, we see that the positive integers and the square numbers must have the ________________________.

Bijections With Infinite Sets:

In Hilbert's hotel, when an infinite amount of new guests arrived, we asked the infinite amount of guests in the hotel rooms to move to the room number that is two times their current room number. This suggests that just as many even positive integers exist as positive integers, since the infinite amount of people can fit in the rooms with positive integers and the rooms with just even positive integers. How can we show that the same amount of even-numbered hotel rooms exist as the amount of hotel rooms labelled with positive integers (both odd and even numbers)?

Types of Infinity

We have discussed that infinity is a concept that means "bigger than anything else." Can we have different *types* of infinity? Yes, we can. There are two main types of infinity, which we call **countable** and **uncountable** infinity.

Countable Infinity

Countable Infinity: The type of infinity that we ____________ theoretically ____________ if we had an infinite amount of time. We could count it, it would just take forever. We consider the ____________ of an infinite set to be ________________________ if we can make a ____________ between the ____________ and said set. Therefore, our previous bijection with the square numbers and the positive integers and our bijection with the even positive integers and positive integers tell us that the cardinalities (or number of elements) of these sets are all countably infinite.

<u>Exercise:</u> How would you count from 0 to 1, counting every single real (decimal) number between 0 and 1? What comes after 0?

Uncountable Infinity

Uncountable Infinity: If you ____________ make a ____________ between the ____________ integers and an infinite set, then the ____________ of said set is ________________________. In this case, the set contains too many elements to be countable.

So far, we know that a countably infinite number of positive integers exist, a countably infinite number of even positive integers exist, and a countably infinite number of square numbers exist. What about the real numbers?

Real numbers include not just all integers (...-2, -1, 0, 1, 2...), not just all fractions, but also all decimals that can't be expressed as fractions.

Try for a second to imagine how you could count all the real numbers. You can have decimals such as 0.222, 0.22222, 0.2222222222, and you can add as many 2s onto the end as you want all the way up until you have infinitely many 2s. Every time you add just one 2, you get a different number. This is still a very small fraction of possible real numbers that you can have.

You can also have 0.1222, 0.122222, 0.2122, 0.21222222 etc. There are infinitely many ways that you can write different decimal numbers using just the digit 2, there are also infinitely many ways that you could write different decimal numbers using just the digit 2 and using the digit 1 once, and there are also infinitely many ways that you could write different decimal numbers using just the digit 2 and using the digit 1 twice.

Even this list that I am making of different scenarios where you can make an infinite amount of decimal numbers could go on infinitely.

<u>Exercise:</u>

Write out 5 decimal numbers with 5 decimal places using any numbers you wish, just make sure each number is different. Circle the diagonal digits:

Write out a new decimal number with all of your digits from this diagonal:

Now change each individual digit after your decimal:

Was this number in your original list of 5 numbers?

The real numbers do not seem as easy to count as the integers, or the squares now, do they? There is no possible way that you could even hope to count all of the real numbers, even with an infinite amount of time. You could also not make a bijection between the positive integers and real numbers. We will show why with Cantor's Diagonal on the next page. So, we have an ____________ infinite amount of real numbers. Uncountable infinities are bigger than countable infinities, hence more real numbers exist than square numbers or positive integers.

Cantor's Diagonal

The following information is from mathigon.org. Video

$$
\begin{array}{cccccccc}
x_1 & . & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & \ldots \\
x_2 & . & x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & \ldots \\
x_3 & . & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & \ldots \\
x_4 & . & x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & \ldots \\
\ldots & & & & & & &
\end{array}
$$

Suppose that in this grid, every real number that exists is listed. So, every row creates a decimal number, where the rows and columns go on infinitely.

Using all of the digits in the diagonal (digits in red), we can create the number,

$$x \quad . \quad x_{11} \quad x_{22} \quad x_{33} \quad x_{44} \quad \ldots$$

If we change each of the digits in this number, we get a new number that was not in our grid,

$$x \quad . \quad y_{11} \quad y_{22} \quad y_{33} \quad y_{44} \quad \ldots$$

We know that this number was not listed in our grid because we changed one digit from every row when we changed the digits in the diagonal. This means that our new number differs from every single real number in our infinitely long grid by at least one digit, so our number is not in the grid.

This means that we cannot possibly list out all of the real numbers that exist, which is the same as saying we cannot create a bijection between the positive integers and the real numbers.

Cantor's diagonal demonstrates the idea of uncountable infinity. There are an infinite number of real numbers and an infinite number of positive integers, but the cardinality of the ____________ numbers is ____________ than the cardinality of the ____________.

Exercise:

Try it again! Write out 10 decimal numbers with 10 decimal places using any numbers you wish, just make sure each number is different. Circle Cantor's diagonal:

Write out a new decimal number with all of your digits from Cantor's diagonal:

Now change each individual digit after your decimal:

You now have a new number that was not in your original list.

Cardinalities and the Continuum Hypothesis

The symbol we use for any set with a cardinality of countable infinity is $\aleph_0$ (____________). Uncountable infinity however, can be broken down into different cardinalities, each bigger than the next.

The Continuum Hypothesis is the idea that there is __________________________ that is between $\aleph_0$ and the cardinality of the real numbers. It is the idea that the cardinality of the real numbers is the next biggest cardinality after $\aleph_0$. The continuum hypothesis is actually mathematically ____________. They've actually proved that they *can't* prove or disprove it!

Problem Set

* Indicates challenge problems.

1. You are gifted an infinite amount of pokemon cards from one friend and 335 pokemon cards from another friend for your birthday. How many pokemon cards do you have in total?

2. It takes you an infinite amount of time to write part 1 of your math test and an infinite amount of time to write part 2 of your math test. How long did it take you to complete your entire math test, all together?

3. I rip the last 1999 pages out of my book that has infinitely many pages. How many pages are left in the book?

4. You have five baskets. Each basket contains an infinite number of strawberries. How many strawberries do you have in total?

5. Pretend you proved that an infinite number of universes existed. Further, pretend you proved for certain that every universe was infinite in size (ie. took up an infinite amount of space). How much space, in total, would every universe take up?

6. List two different examples of sets of numbers whose cardinality is countably infinite.

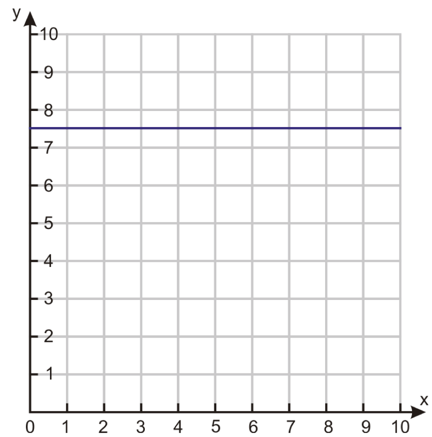
7. Create a bijection to show that we have the same number of odd positive integers as we do positive integers.

8. Is the cardinality of real numbers between 0 and 1 countably or uncountably infinite? Why?

9. Write out a list of 15 numbers, each with 15 decimal places. Circle Cantor's diagonal, then make a decimal number with the numbers in that diagonal and change each of the individual digits. Did you end up with a new number that was not in your list? (This is the same as the exercise we did in class, only longer).

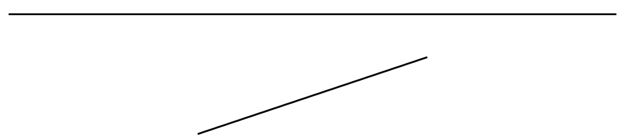
10. *

(a) How many points are on the following line:



Retrieved from www.ck12.org

(b) Which of the following lines has more points?



(c) Which will have more points: A curve with finite length, or a line with infinite length?