UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

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To Infinity, and Beyond!

What is Infinity?

Retrieved from: www.edgemagazine.net

Infinity represents a quantity with no bounds. In math, we write infinity as “∞” and it represents something that is bigger than any other number. However, infinity is not an actual number, but rather a concept. That is to say, it does not behave the same as a number would.

In our warm-up, we thought of numbers that got successively bigger. This list of numbers can go on forever and there is no one value that we can say is the “biggest.” So, we say that an infinite amount of numbers exist.

Exploring Infinity as a Concept

Buzz Lightyear is on a mission to travel to the edge of the universe. He sets off flying at the speed of light. He travels past Saturn, Neptune and out of our solar system. He passes out of our Milky Way galaxy and buzzes past millions more. When will he get to the edge of the universe?

Retrieved from: www.pinterest.ca

Well, we don't know for sure. However, many physicists think that the answer to this question is ___“never,”___ as many believe that the universe is ___infinite___ in size. That is to say, the universe may go on ___forever___ in size. If this is true, Buzz Lightyear will ___never___ reach the edge of the universe, because ___no such thing would exist___.

In order to imagine the size of the universe, you must allow yourself to imagine the largest thing you can possibly think of, imagine the universe taking up as much space as you can possibly think of - the universe is still even bigger than that.

Retrieved from: flamemedia.tv

Try to answer the following questions:

1. (a) An infinite number of people stand in line at Walmart. If one more person joins the line, how many people are now in line?

   If we add one person to an infinite number of people, there are still an infinite number of people in line!

   (b) If 200 more people join the end of the line, how many people are now in line?

   If we add 200 people to an infinite number of people, we still have an infinite number of people in line!

   (c) If an infinite number of people are also standing in line at Sobeys but Sobeys closes and everyone in this line joins the end of the Walmart line with an infinite amount of people in it, how many people are now in this line?

   Surprise! When we add an infinite number of people to an infinite number of people, we still have an infinite number of people in line!

2. You have a lot of spare time one weekend and decide to bake an infinite number of cookies. If your friend steals 3 cookies from you, how many cookies do you have now?

   Good news! When we take away a finite number of cookies from an infinite number of cookies, we still have an infinite amount of cookies! So, if your friend takes three of your infinite amount of cookies, you still have an infinite amount of cookies.

3. (a) If I invite an infinite number of guests to my birthday party and each guest brings two of their friends, how many people are at my birthday party?

   If I have an infinite amount of people, each bringing two people, I have an infinite amount of groups of two. It turns out I still will have an infinite amount of people at my party!

   (b) If I invite an infinite number of guests to my birthday party and each guest brings an infinite number of their friends, how many people are at my birthday party?

Now I have an infinite amount of groups of infinity...I still have an infinite amount of people at my party!

None of this makes sense? Let's try demonstrating some of these examples.

Hilbert's Hotel

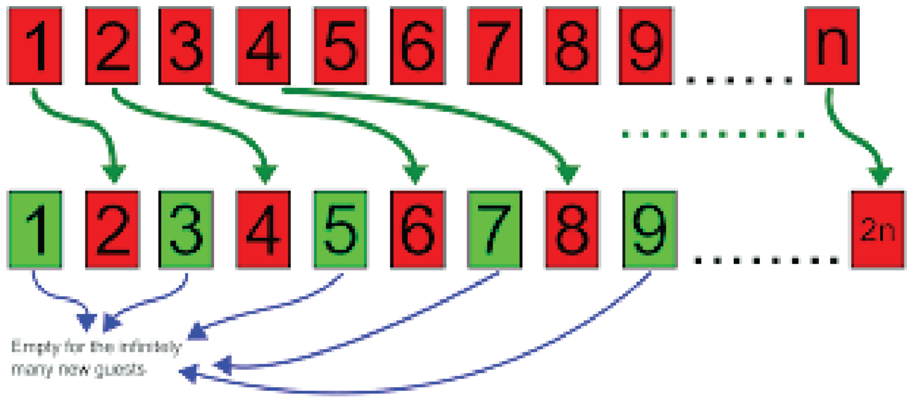
Video

Retrieved from: intuitivescienceblog.wordpress.com

The Hilbert Hotel has an infinite amount of rooms. You and your friend go to check in to the Hilbert Hotel, but you are told that all of the rooms are full. Your friend starts to turn away, but you realize something and stop them. If the hotel has an infinite number of rooms, you can ask the people in each room to move to the next room. Everyone in room 1 would move to room 2, the people in room 2 would move to room 3, and so on forever. This would leave room number 1 free for you and your friend.

What would happen if an *infinite* amount of people wanted to check into the Hilbert Hotel, but the hotel was full?

Following what we did before, we would ask everyone to move up an infinite amount of rooms, to make an infinite amount of new rooms for the infinite amount of people who just arrived. However, this does not make sense because you cannot ask someone to move up an infinite amount of rooms, as infinity is not a real number and they would not know which room to move to. Instead, we can ask everyone to move to the room number that is twice their current room number. This means that the person in room 1 would move to room 2, the people in room 2 would move to room 4, the people in room 3 would move to room 6 and so on infinitely. Since twice any number is an even number, we now have all the odd numbers left, but because there is an infinite amount of odd numbers we now have an infinite amount of odd numbered rooms for the infinite amount of new people to go to.



Retrieved from en.wikipedia.org

Cardinality

Cardinality: The number of ___elements___ in a _____set_____.

Examples:

1. If I have a basket (or set) of 12 apples, what is the cardinality of this set?

   Elements = Apples
   Number of elements/apples = 12
   Therefore, the cardinality of this set is 12, because there are 12 elements.

2. Determine the cardinality of the following set of numbers: $\{1, 3, 5, 7, 9\}$.

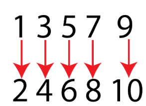
   Each number in this set represents one element.
   Amount of numbers/elements in this set = 5.
   So, since we have 5 elements in this set, the cardinality of the set is 5.

Bijections

Bijection: There is a bijection between two sets when we can line up the two sets and link every element of each set with exactly one element of the other. This means that every element in each set is paired with exactly one element in the other, with no one element in either set left without a partner. If we can find a bijection between two sets, we say that two sets have the same cardinality (the same number of elements).

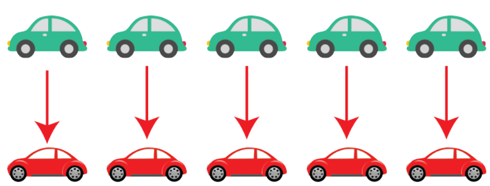
Example of a Bijection:

Show that the following two sets have the same cardinality: $\{1, 3, 5, 7, 9\}$, $\{2, 4, 6, 8, 10\}$

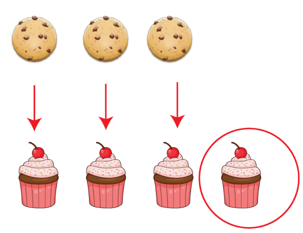


Exercise:

Which of the following are bijections:

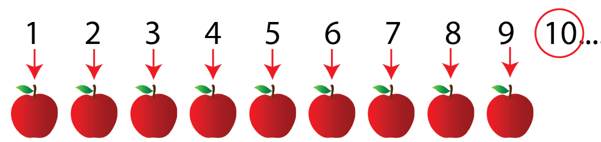
1. 

We can pair every car in each set with exactly one car in the other, hence this is a bijection.

2. 

Not every cupcake can be paired with a cookie, we have one lone cupcake left. Hence, this is not a bijection.

3. I have 9 apples in my bag and I want to prove that the same number of positive integers exist as the number of apples that I have. How do I do this?



The number of apples in my bag is a finite value, since I could assign a number to it, 9. However, the amount of positive integers that exist is infinite, it keeps going forever. So, we clearly do not have the same amount of apples as we do positive integers. This is shown in the bijection because we were not able to pair each positive integer with an apple. Once we get to 10, we run out of apples and can no longer make any more pairs.

Infinite Sets

Infinite Set: Any set that has an infinite number of elements.

Examples of infinite sets include...

- The positive integers $\{1, 2, 3, 4, ...\}$
- All integers $\{... -2, -1, 0, 1, 2, ...\}$
- Even numbers
- Odd numbers
- Square numbers $\{1, 4, 9, 16, 25, ...\}$

How can we compare the sizes of infinite sets?
Just like with our finite sets, we can use bijections to show that two infinite sets have the same number of elements.

Galileo's Paradox

Galileo said that there are just as many perfect square numbers as there are positive integers.

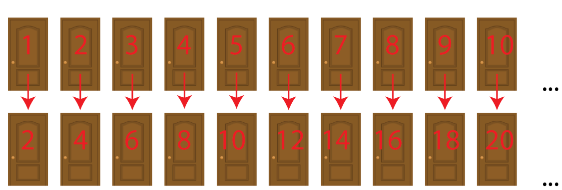
Galileo used a bijection to prove his claim:

1	2	3	4	5	6	7	8	9	10...
↓	↓	↓	↓	↓	↓	↓	↓	↓	↓
1	4	9	16	25	36	49	64	81	100...

Here, we see that we are able to link each positive integer with a square number (1 and 1, 2 and 4, 3 and 9, etc.). We can just square each positive integer and get the corresponding perfect square number. Also, each perfect square links to one positive integer, we can square root the perfect square to get the positive integer. We can see that this will go on forever and we will always be able to pair each element in the first infinite set with exactly one element of the second, with no element left over in either set. Because we were able to create a bijection, we see that the positive integers and the square numbers must have the same cardinality.

Bijections With Infinite Sets:

In Hilbert's hotel, when an infinite amount of new guests arrived, we asked the infinite amount of guests in the hotel rooms to move to the room number that is two times their current room number. This suggests that just as many even positive integers exist as positive integers, since the infinite amount of people can fit in the rooms with positive integers and the rooms with just even positive integers. How can we show that the same amount of even-numbered hotel rooms exist as the amount of hotel rooms labelled with positive integers (both odd and even numbers)?



We can create a bijection by pairing up every even-numbered door with a door labelled with a positive integer (1 and 2, 2 and 4, 3 and 6, n and $2n$, etc.), this bijection shows us that we have the same number of each type of door.

Types of Infinity

We have discussed that infinity is a concept that means "bigger than anything else." Can we have different *types* of infinity? Yes, we can. There are two main types of infinity, which we call **countable** and **uncountable** infinity.

Countable Infinity

Countable Infinity: The type of infinity that we <u>could</u> theoretically <u>count</u> if we had an infinite amount of time. We could count it, it would just take forever. We consider the <u>cardinality</u> of an infinite set to be <u>countably infinite</u> if we can make a <u>bijection</u> between the <u>positive integers</u> and said set. Therefore, our previous bijection with the square numbers and the positive integers and our bijection with the even positive integers and positive integers tell us that the cardinalities (or number of elements) of these sets are all countably infinite.

<u>Exercise:</u> How would you count from 0 to 1, counting every single real (decimal) number between 0 and 1? What comes after 0?

Uncountable Infinity

Uncountable Infinity: If you <u>cannot</u> make a <u>bijection</u> between the <u>positive</u> integers and an infinite set, then the <u>cardinality</u> of said set is <u>uncountably infinite</u>. In this case, the set contains too many elements to be countable.

So far, we know that a countably infinite number of positive integers exist, a countably infinite number of even positive integers exist, and a countably infinite number of square numbers exist. What about the real numbers?

Real numbers include not just all integers (...-2, -1, 0, 1, 2...), not just all fractions, but also all decimals that can't be expressed as fractions.

Try for a second to imagine how you could count all the real numbers. You can have decimals such as 0.222, 0.22222, 0.2222222222, and you can add as many 2s onto the end as you want all the way up until you have infinitely many 2s. Every time you add just one 2, you get a different number. This is still a very small fraction of possible real numbers that you can have.

You can also have 0.1222, 0.122222, 0.2122, 0.21222222 etc. There are infinitely many ways that you can write different decimal numbers using just the digit 2, there are also infinitely many ways that you could write different decimal numbers using just the digit 2 and using the digit 1 once, and there are also infinitely many ways that you could write different decimal numbers using just the digit 2 and using the digit 1 twice.

Even this list that I am making of different scenarios where you can make an infinite amount of decimal numbers could go on infinitely.

<u>Exercise:</u>

Write out 5 decimal numbers with 5 decimal places using any numbers you wish, just make sure each number is different. Circle the diagonal digits:

Write out a new decimal number with all of your digits from this diagonal:

Now change each individual digit after your decimal:

Was this number in your original list of 5 numbers?

The real numbers do not seem as easy to count as the integers, or the squares now, do they? There is no possible way that you could even hope to count all of the real numbers, even with an infinite amount of time. You could also not make a bijection between the positive integers and real numbers. We will show why with Cantor's Diagonal on the next page. So, we have an <u>uncountably</u> infinite amount of real numbers. Uncountable infinities are bigger than countable infinities, hence more real numbers exist than square numbers or positive integers.

Cantor's Diagonal

The following information is from mathigon.org. Video

$$\begin{array}{cccccccc} x_1 & . & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & \dots \\ x_2 & . & x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & \dots \\ x_3 & . & x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & \dots \\ x_4 & . & x_{41} & x_{42} & x_{43} & x_{44} & x_{45} & \dots \\ \dots & & & & & & & \end{array}$$

Suppose that in this grid, every real number that exists is listed. So, every row creates a decimal number, where the rows and columns go on infinitely.

Using all of the digits in the diagonal (digits in red), we can create the number,

$$x \quad . \quad x_{11} \quad x_{22} \quad x_{33} \quad x_{44} \quad \dots$$

If we change each of the digits in this number, we get a new number that was not in our grid,

$$x \quad . \quad y_{11} \quad y_{22} \quad y_{33} \quad y_{44} \quad \dots$$

We know that this number was not listed in our grid because we changed one digit from every row when we changed the digits in the diagonal. This means that our new number differs from every single real number in our infinitely long grid by at least one digit, so our number is not in the grid.

This means that we cannot possibly list out all of the real numbers that exist, which is the same as saying we cannot create a bijection between the positive integers and the real numbers.

Cantor's diagonal demonstrates the idea of uncountable infinity. There are an infinite number of real numbers and an infinite number of positive integers, but the cardinality of the ___real___ numbers is ___bigger___ than the cardinality of the positive integers.

Exercise:

Try it again! Write out 10 decimal numbers with 10 decimal places using any numbers you wish, just make sure each number is different. Circle Cantor's diagonal:

Write out a new decimal number with all of your digits from Cantor's diagonal:

Now change each individual digit after your decimal:

You now have a new number that was not in your original list.

Cardinalities and the Continuum Hypothesis

The symbol we use for any set with a cardinality of countable infinity is $\aleph_0$ (__Aleph 0__). Uncountable infinity however, can be broken down into different cardinalities, each bigger than the next.

The Continuum Hypothesis is the idea that there is ___no cardinalty___ that is between $\aleph_0$ and the cardinality of the real numbers. It is the idea that the cardinality of the real numbers is the next biggest cardinality after $\aleph_0$. The continuum hypothesis is actually mathematically __unprovable__. They've actually proved that they *can't* prove or disprove it!

Problem Set

* Indicates challenge problems.

1. You are gifted an infinite amount of pokemon cards from one friend and 335 pokemon cards from another friend for your birthday. How many pokemon cards do you have in total?

   Adding 335 cards to an infinite number of cards still gives you an infinite number of cards.

2. It takes you an infinite amount of time to write part 1 of your math test and an infinite amount of time to write part 2 of your math test. How long did it take you to complete your entire math test, all together?

   Adding an infinite amount of time to an infinite amount of time...still gives you an infinite amount of time.

3. I rip the last 1999 pages out of my book that has infinitely many pages. How many pages are left in the book?

   If I take out a finite number of pages from a book with infinitely many pages...I still have infinitely many pages.

4. You have five baskets. Each basket contains an infinite number of strawberries. How many strawberries do you have in total?

   You still have an infinite amount of strawberries in total.

5. Pretend you proved that an infinite number of universes existed. Further, pretend you proved for certain that every universe was infinite in size (ie. took up an infinite amount of space). How much space, in total, would every universe take up?

   You have an infinite number of universes, each taking up infinite space. In total, they would take up infinite space.

6. List two different examples of sets of numbers whose cardinality is countably infinite.

Possible answers include: The amount of positive integers, all integers, square numbers, even numbers, odd numbers etc.

7. Create a bijection to show that we have the same number of odd positive integers as we do positive integers.

$$\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\ldots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19\ldots \end{array}$$

Here, we see that we are able to link each odd positive integer to a positive integer (positive integer n is matched with odd positive integer $2n - 1$). Therefore, since a bijection exists between these two sets of numbers, the cardinality of both sets is the same. That is, the same number of odd numbers exist as positive integers.

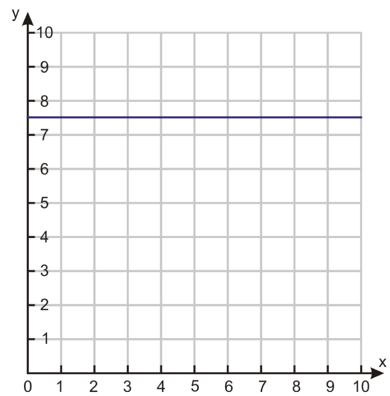
8. Is the cardinality of real numbers between 0 and 1 countably or uncountably infinite? Why?

The cardinality is uncountably infinite.
There are an infinite number of decimal numbers that are between 0 and 1 (ex. 0.00001, 0.6374827 etc.). You can recreate Cantor's diagonal, only using the digit zero before the decimal, to demonstrate this idea.

9. Write out a list of 15 numbers, each with 15 decimal places. Circle Cantor's diagonal, then make a decimal number with the numbers in that diagonal and change each of the individual digits. Did you end up with a new number that was not in your list? (This is the same as the exercise we did in class, only longer).

10. *

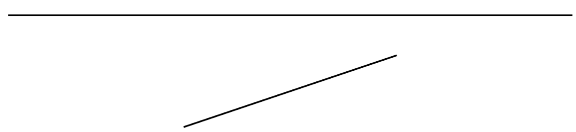
    (a) How many points are on the following line:



Retrieved from www.ck12.org

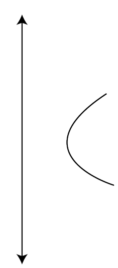
There are an uncountably infinite amount of points on the line. Every point on this line can be described as a set of coordinates of real numbers, like (0,7.5), (3,7.5) etc. Since the points on the line can be described as real numbers, which are uncountably infinite, we can have an uncountably infinite number of points on the line.

(b) Which of the following lines has more points?



The amount of points on each line is uncountably infinite, meaning that the sets of points on each line have the same cardinality.

(c) Which will have more points: A curve with finite length, or a line with infinite length?



An infinitely long line has an uncountably infinite amount of points on it, but a finite curve also has an uncountably infinite amount of points on it. So, the sets of points on each line/curve have the same cardinality.