UNIVERSITY OF
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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

March 3/4 2020

Topology

Introduction

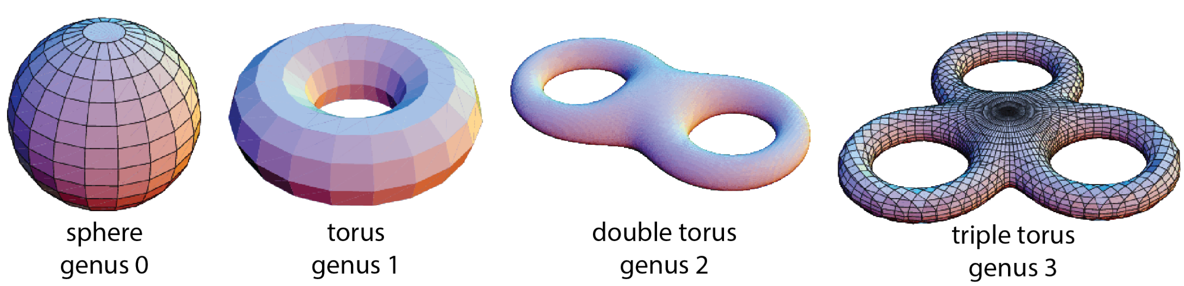
Topology is a branch of math which studies what happens when objects are stretched, twisted, and deformed, but not cut.

Equivalence

In topology, two objects are equivalent if one can be obtained by deforming (but not cutting) the other. For example, if I use clay to make a sphere, I can then deform it into a cube without tearing any holes in it. Thus, a sphere and a cube are **topologically equivalent** or **homeomorphic**.

On the other hand, if I have a clay sphere and I want to make a doughnut, I have to cut a hole in the centre, so a sphere and a doughnut are not homeomorphic.

The number of holes a surface has is called the **genus** of a surface.

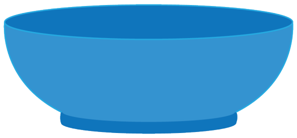


Images retrieved from `http://mathworld.wolfram.com/`

Deformations

Let's watch this video to see how topologists think of deformations: `https://www.youtube.com/watch?v=k8Rxep2Mkp8`

Exercise 1. Find the genus of each surface:

A


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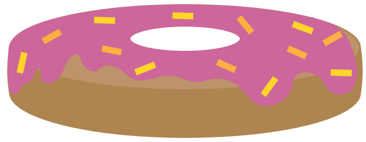

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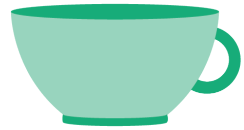

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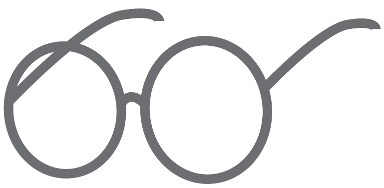

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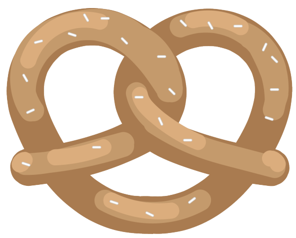

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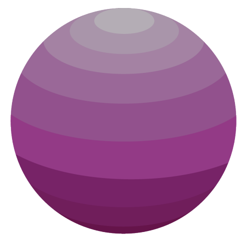

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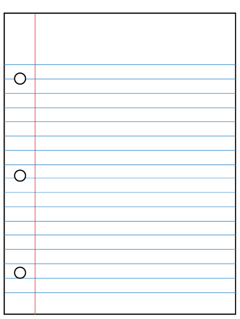

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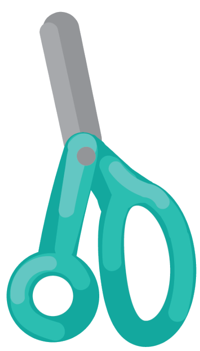

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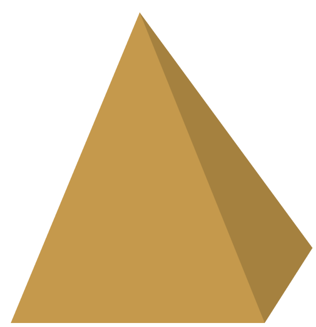
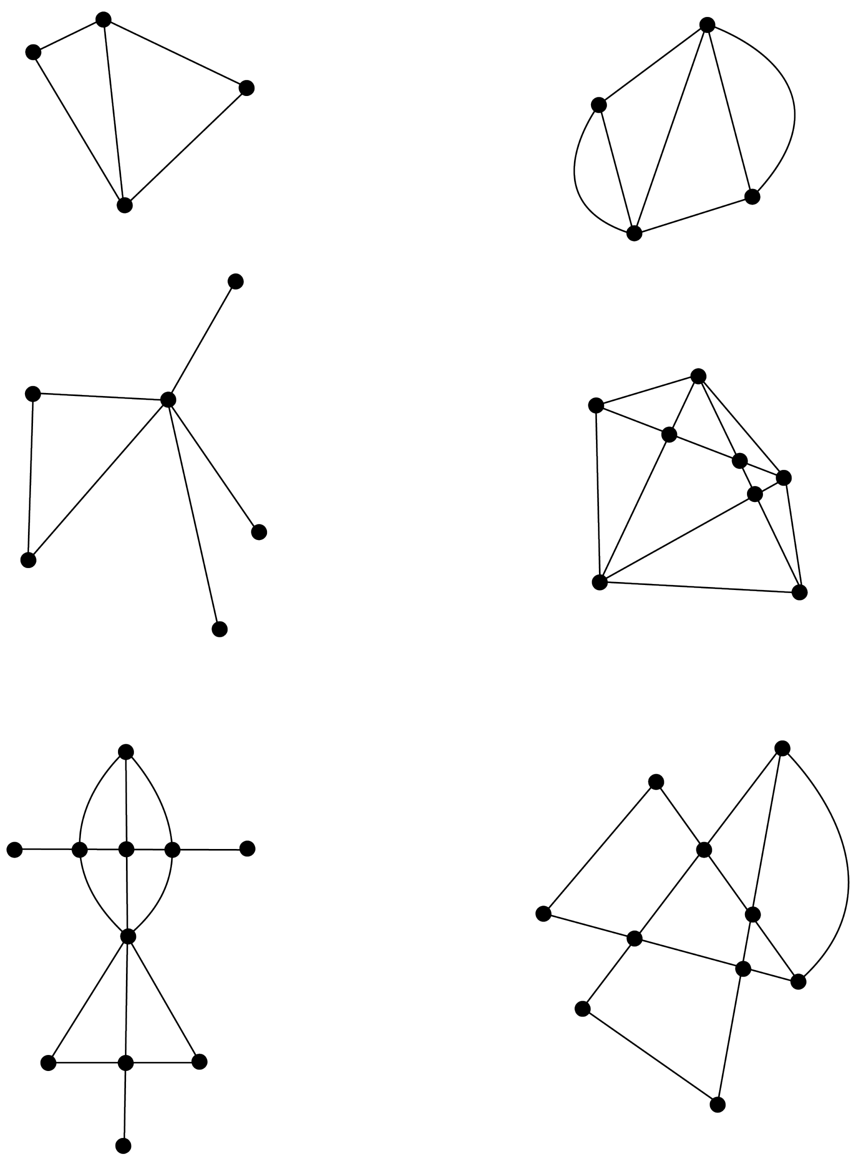


Euler Characteristic

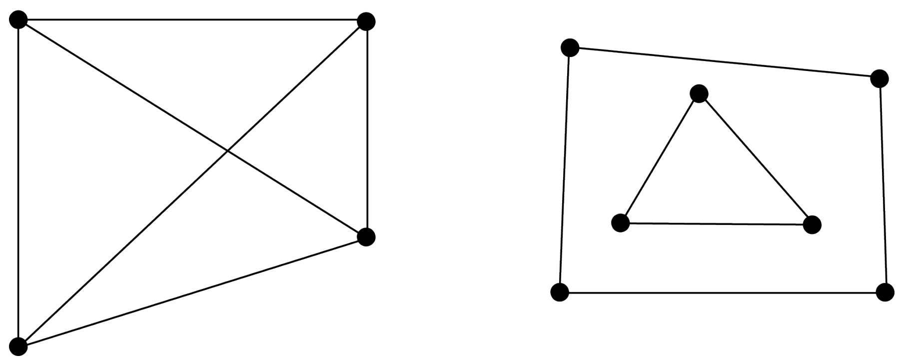
The following section is based on the booklet *Graph Theory for Kids* by Joel David Hamkins.

Let's look at some graphs! In topology and graph theory, a **graph** is a collection of points (called **vertices**) and lines connecting these points (called **edges**). Leonhard Euler, one of the most famous mathematicians of the 18th century, discovered a very interesting property of graphs.

Exercise 2. The Euler characteristic (χ) = # of vertices - # of edges + # of regions
Calculate the value of χ for each of the graphs below. What do you notice?



Exercise 3. Calculate the Euler characteristic for these examples. What is different about these graphs than the previous ones?



The graphs in Exercise 2 are called **connected planar graphs**. They do not have edges that ________________________ and they do not have ________________________ pieces.

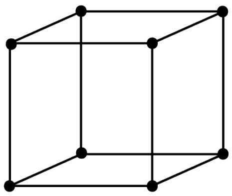
Exercise 4. Turn the graphs from Exercise 3 into connected planar graphs and re-calculate their Euler characteristic.

Every connected planar graph has an Euler characteristic of ________________________!

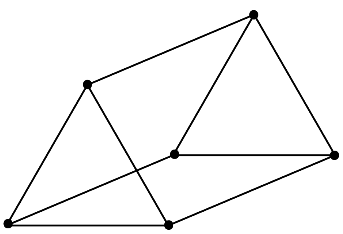
We can generalize the Euler characteristic to 3D objects: just replace regions with faces of the object.

$$\chi = V - E + F$$

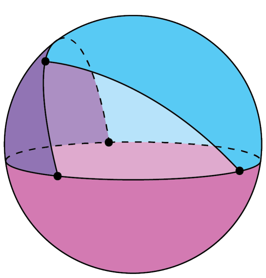
Exercise 5. Calculate the Euler characteristic of each surface:

1. 

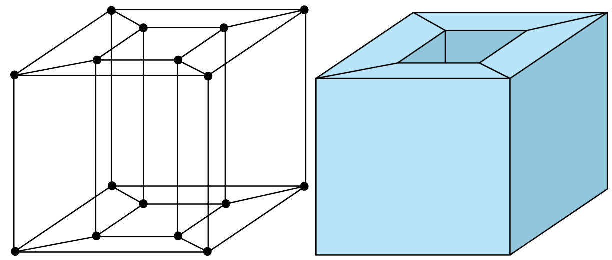
cube

2. 

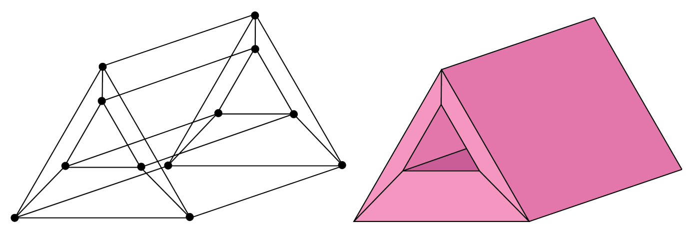
triangular prism

3. 

sphere

4. 

cube with a hole

5. 

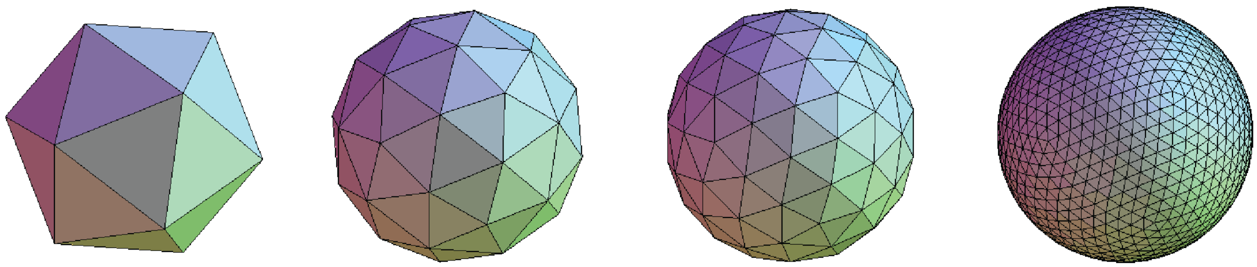
triangular prism with a hole

Look at the number of holes each surface has and its Euler characteristic. Do you notice any patterns?

It turns out, all homeomorphic surfaces have the same Euler characteristic, so we can deform a shape as much as we would like without cutting it, and the Euler characteristic will stay the same. Properties like this, which remain unchanged after applying certain types of transformations, are called **invariants**.

Did you notice the way in which we divided up the sphere into triangles? This is called **triangulation**.

Based on what we discovered, what is the Euler characteristic of the following four shapes?



Images by Theon, retrieved from `https://en.wikipedia.org/wiki/User:Tomruen/Geodestic_sphere`

Computer modelling softwares represent curved-surface models with triangulations. This is how movies are animated and 3D models are built. As the triangles get smaller and smaller, they look more and more like a smooth curved surface! Why do you think computers prefer triangulations to real curved surfaces?

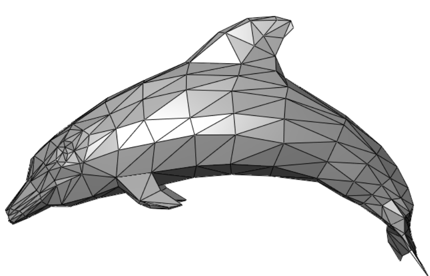
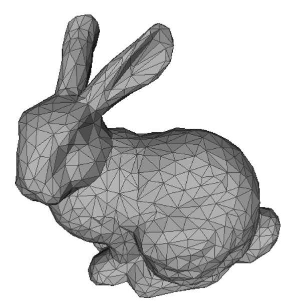


Image by Chrschn, retrieved from `https://commons.wikimedia.org/wiki/File:Dolphin_triangle_mesh.png`

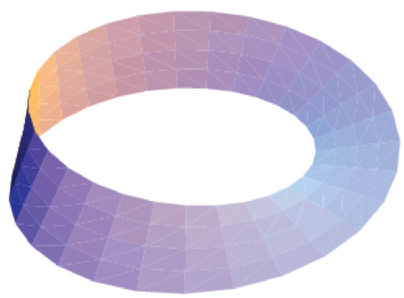


Retrieved from `http://www.lix.polytechnique.fr/~maks/Verona_MPAM/TD/TD2/`

Möbius Strips

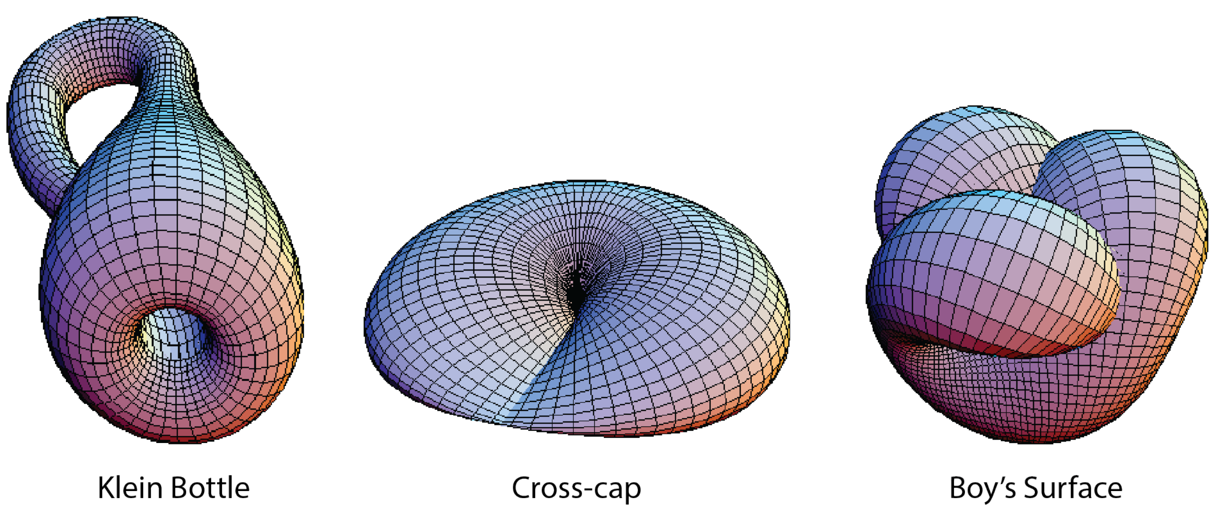
If I give you a strip of paper and some tape, what can you do to it so that you can trace your finger along both sides of it without ever crossing an edge?

The solution to this puzzle is a very important shape in topology, called the **Möbius strip**. It is a strange surface: it is 3D but only has 1 side! To create it, connect the two ends of your paper as if you were going to make a loop, but twist one end 180° before gluing it to the other end.



Retrieved from `http://mathworld.wolfram.com/MoebiusStrip.html`

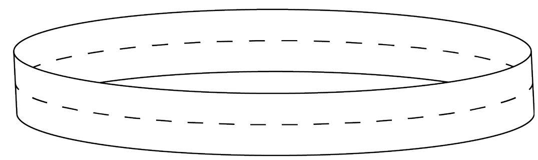
The Möbius strip does not have a distinguishable "inside" and "outside" like other 3D shapes. There is no way to consistently define an "orientation" everywhere on the strip, so it is called a **non-orientable** surface. The following are 3D approximations of some other non-orientable surfaces (we cannot really represent these without having 4 dimensions):



Images by AugPi, retrieved from `https://commons.wikimedia.org`

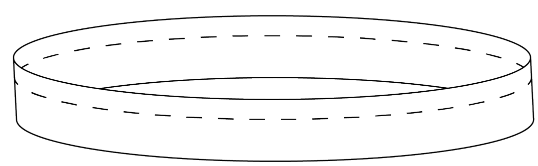
Let's say I have two identical strips of paper. They are each 1 cm thick and 10 cm long. I turn one strip into a loop (0 twists) and the other into a Möbius strip (1 twist).

If I cut along the $\frac{1}{2}$ point of the loop's width, what will I get?



What do you think will happen if I do the same with the Möbius strip?

What if I cut along the $\frac{1}{3}$ point of the loop's width?

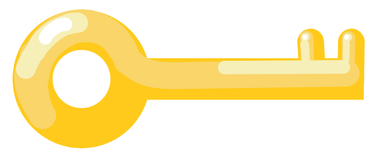


What if I do the same with the Möbius strip?

Problem Set

Problems marked with an asterisk (*) are challenge questions.

1. Calculate the genus of the following objects:

(a)  (b)  (c)  (d) 

2. Which of the objects above are homeomorphic?

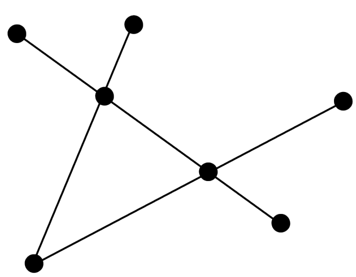
3. Come up with two homeomorphic objects of your own.

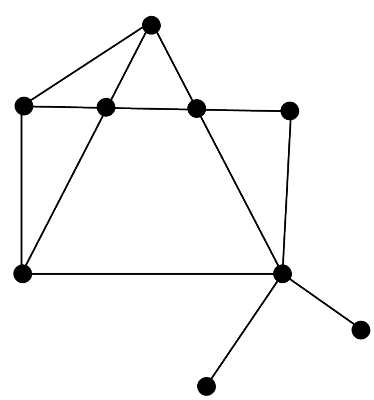
4. (a) What is the genus of a paper loop?

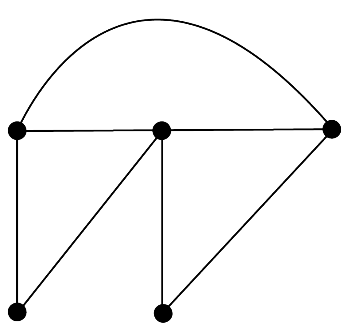
   (b) What is the genus of a Möbius strip?

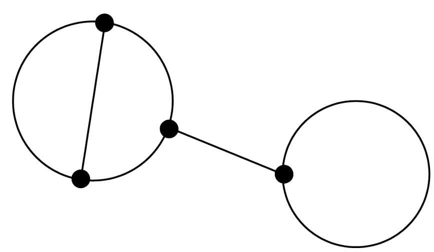
   (c) * Are a paper loop and a Möbius strip homeomorphic? *(hint: what would you have to do to transform one into the other?)*

5. Calculate the Euler characteristic of the following graphs:







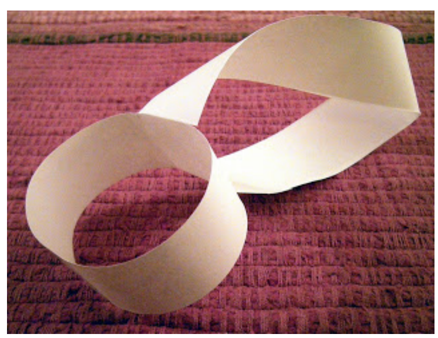


6. This is the universal recycling symbol. Do you notice anything interesting about it?

Retrieved from `https://commons.wikimedia.org/wiki/File:Recycle001.svg`

7. To make a Möbius strip, we took a strip of paper and twisted it (180°) once before gluing the ends together. Make a strip with two twists, and cut along the halfway point of the width like we did with the Möbius strip. What happens? Try it with three twists, or four twists. Do you see a pattern?

8. The following exercise was invented by Martin Gardner. Make a prediction and then try it yourself: Make a cross out of paper. Fold one set of arms into a loop, and the other into a Möbius band like so:

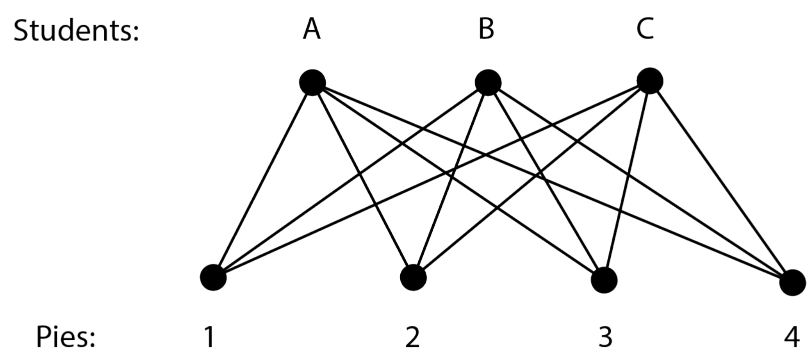


Retrieved from
`http://mathtourist.blogspot.com/2010/10/martin-gardners-mobius-surprise.html`

Cut the Möbius band along the $\frac{1}{3}$ point of its width. Then cut the loop along the $\frac{1}{2}$ point of its width. What happens?

Extension: Graphs, which we looked during this lesson, can be used to simplify certain problems. We can represent objects as vertices and connections between these objects as edges between the vertices.

For example, suppose we have 3 students and 4 pies. If each student wants to take one piece from each pie, how many pieces of pie will be eaten? We can represent this problem using a graph, with each student or pie as a vertex, and edges to represent a student taking a piece of pie.



Counting up the number of edges, we can see that 12 pieces of pie were eaten.

The **degree** of a vertex is the number of vertices it is connected to. One useful theorem for solving graph problems is that if the sum of the degrees of all the vertices of a graph is D, and the number of edges is E, then

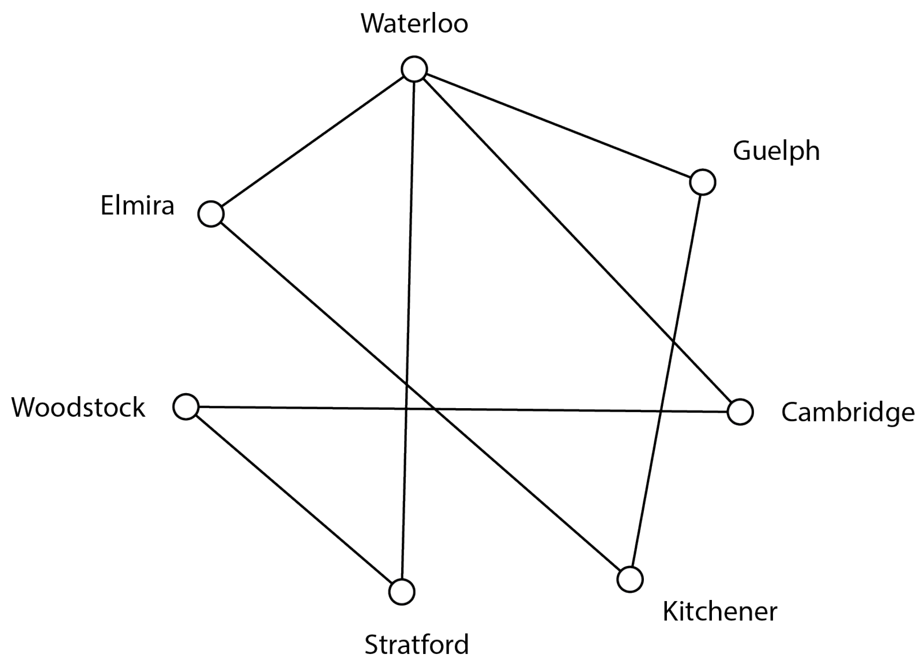
$$D = 2 \times E$$

The remaining questions in the problem set will involve graph theory.

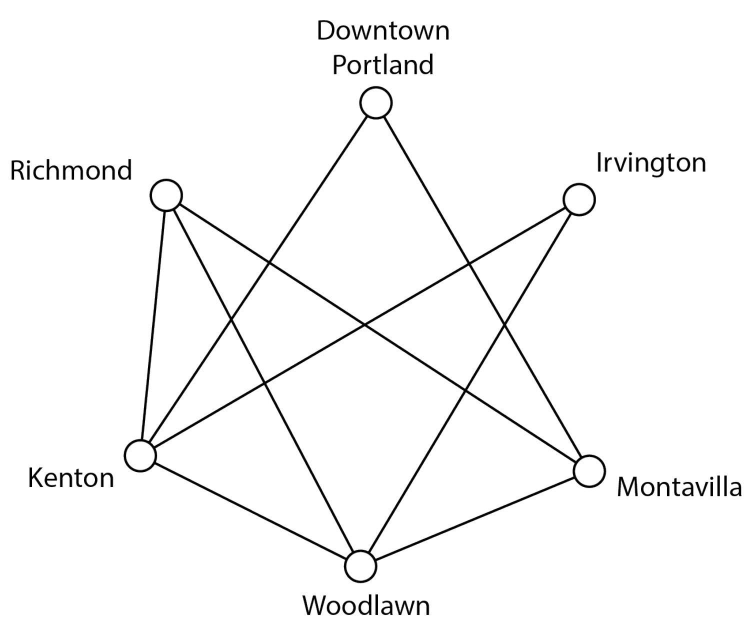
9. (a) * There are 5 people at a party. Is it possible for each of them to know exactly 2 people?

   (b) * Is it possible for each of them to know exactly 3 people?

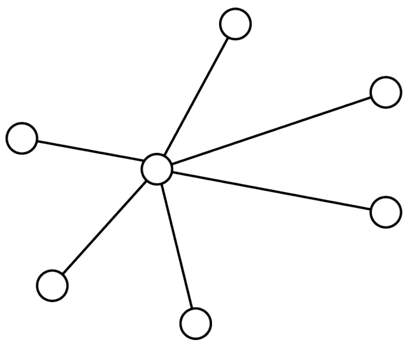
10. * Yara is a cyclist who lives in Waterloo. She wants to start in Waterloo, bike through all of the cities on the following map exactly once, and then return to Waterloo. The map below shows the bike roads available to her. Is it possible?

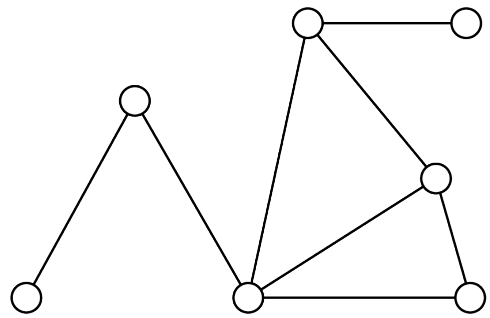


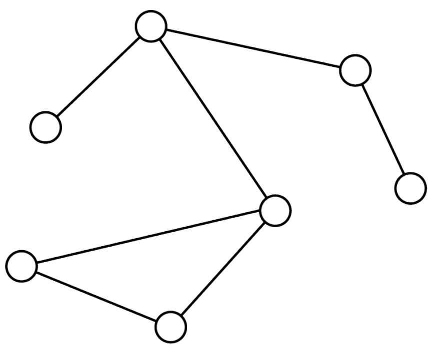
11. * David, Yara's American friend, is also a cyclist. Below is a map of the bike paths available to him. If he wants to start and end his trip in Downtown Portland, and visit each district exactly one time, what route should he take?

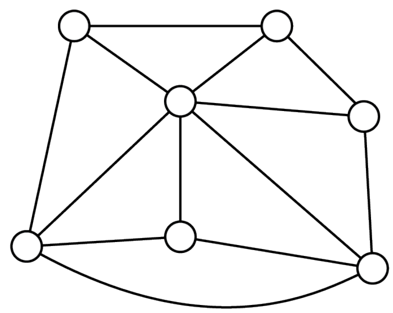


12. * The **chromatic number** of a graph is the number of colours required to colour each vertex such that any vertices connected by an edge are different colours. Find the chromatic number of each graph below:


(a)


(b)


(c)


(d)

13. * Graph colouring (and finding the chromatic number) has a lot of applications. Try to use graph colouring to solve the following problems:

    (a) Suppose there are 7 subjects offered at a school for grade 6 students. Each teacher wants to have an exam during lunch time. However, there are students taking multiple subjects, so classes that share students have to have exams on different days. What is the minimum number of days required to run all 7 exams?

Class	Shares students with ...
Math	Music, Art, Science
Science	Math, Music, English
History	Art, French
English	Science, Music, French
French	Music, History, English
Art	History, Math, Music
Music	Math, Science, English, French, Art

(b) Most maps use different colours for neighbouring regions. Below is a map of Africa. What is the least number of colours required to colour it in so countries that share a border are different colours?

Image by Bruce Jones, retrieved from `http://www.freeusandworldmaps.com`