UNIVERSITY OF WATERLOO

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Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

March 3/4 2020

Topology

Introduction

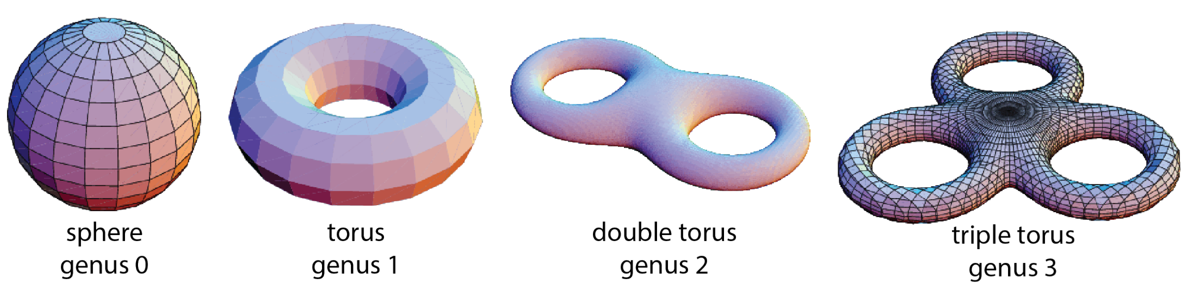
Topology is a branch of math which studies what happens when objects are stretched, twisted, and deformed, but not cut.

Equivalence

In topology, two objects are equivalent if one can be obtained by deforming (but not cutting) the other. For example, if I use clay to make a sphere, I can then deform it into a cube without tearing any holes in it. Thus, a sphere and a cube are **topologically equivalent** or **homeomorphic**.

On the other hand, if I have a clay sphere and I want to make a doughnut, I have to cut a hole in the centre, so a sphere and a doughnut are not homeomorphic.

The number of holes a surface has is called the **genus** of a surface.

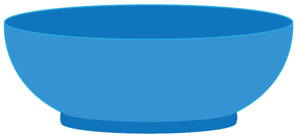


Images retrieved from `http://mathworld.wolfram.com/`

Deformations

Let's watch this video to see how topologists think of deformations: `https://www.youtube.com/watch?v=k8Rxep2Mkp8`

Exercise 1. Find the genus of each surface:

A 

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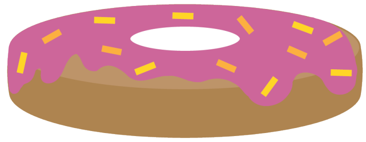
B 

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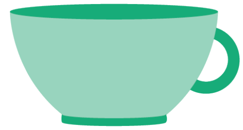
C 

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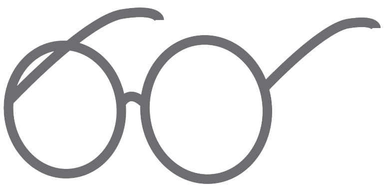
D 

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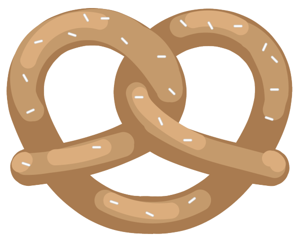
E 

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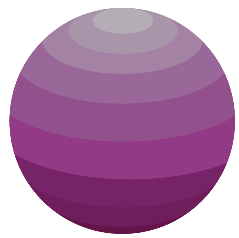
F 

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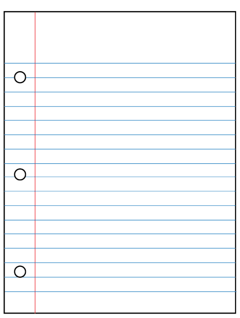
G 

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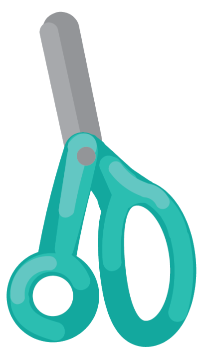
H 

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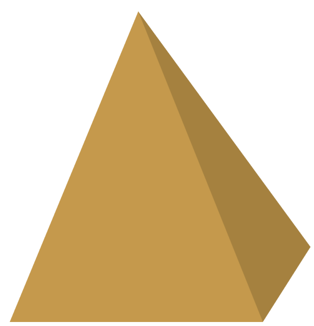
I 

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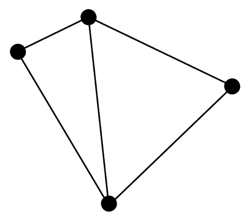
Euler Characteristic

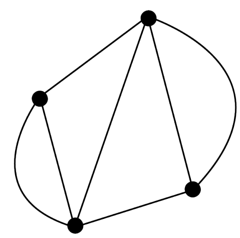
The following section is based on the booklet *Graph Theory for Kids* by Joel David Hamkins.

Let's look at some graphs! In topology and graph theory, a **graph** is a collection of points (called **vertices**) and lines connecting these points (called **edges**). Leonhard Euler, one of the most famous mathematicians of the 18th century, discovered a very interesting property of graphs.

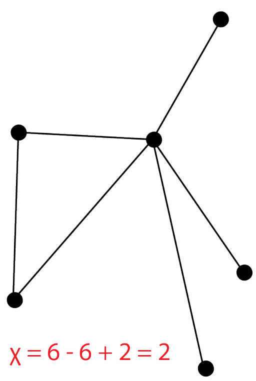
Exercise 2. The Euler characteristic (χ) = # of vertices - # of edges + # of regions
Calculate the value of χ for each of the graphs below. What do you notice?



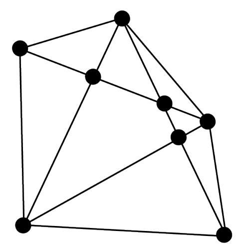
$\chi = 4 - 5 + 3 = 2$



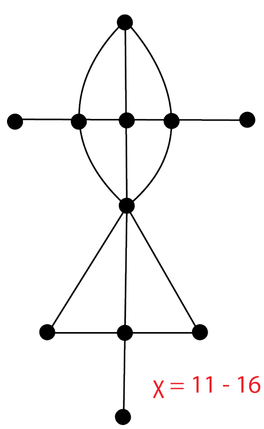
$\chi = 4 - 7 + 5 = 2$



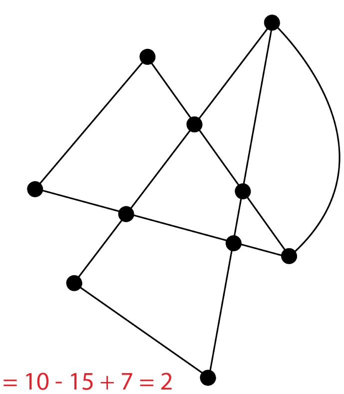
$\chi = 6 - 6 + 2 = 2$



$\chi = 8 - 15 + 9 = 2$

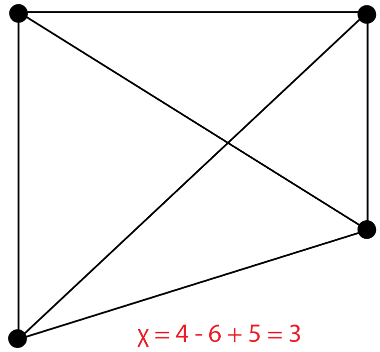


$\chi = 11 - 16 + 7 = 2$

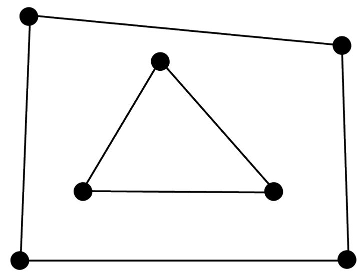


$\chi = 10 - 15 + 7 = 2$

Exercise 3. Calculate the Euler characteristic for these examples. What is different about these graphs than the previous ones?



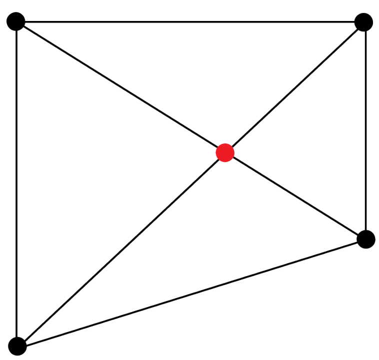
$\chi = 4 - 6 + 5 = 3$



$\chi = 7 - 7 + 3 = 3$

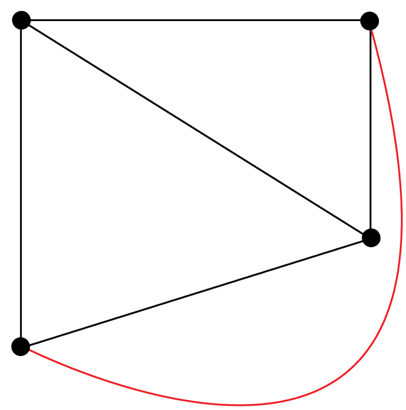
The graphs in Exercise 2 are called **connected planar graphs**. They do not have edges that ______cross______ and they do not have ______disconnected______ pieces.

Exercise 4. Turn the graphs from Exercise 3 into connected planar graphs and re-calculate their Euler characteristic.

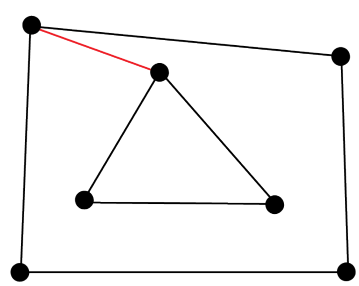


$\chi = 5 - 8 + 5 = 2$

or



$\chi = 4 - 6 + 4 = 2$



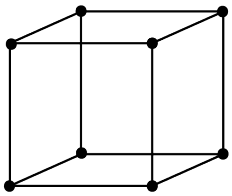
$\chi = 7 - 8 + 3 = 2$

Every connected planar graph has an Euler characteristic of ______2______!

We can generalize the Euler characteristic to 3D objects: just replace regions with faces of the object.

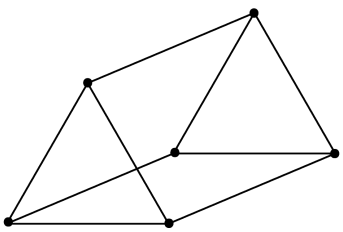
$$\chi = V - E + F$$

Exercise 5. Calculate the Euler characteristic of each surface:

1. 

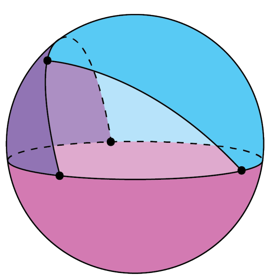
   cube

   $\chi = 8 - 12 + 6 = 2$

2. 

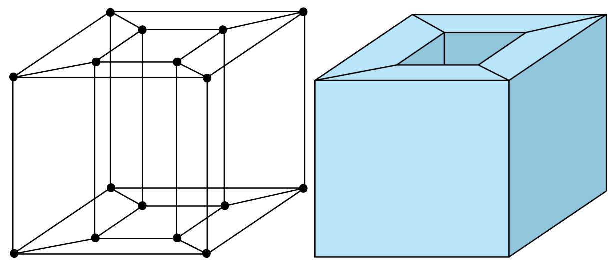
   triangular prism

   $\chi = 6 - 9 + 5 = 2$

3. 

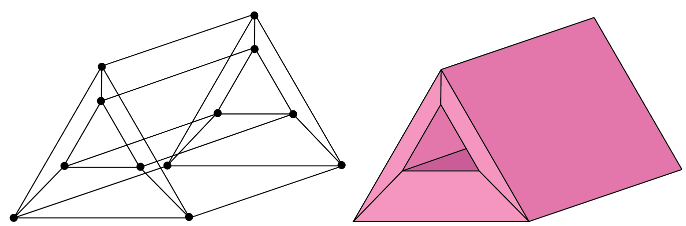
   sphere

   $\chi = 4 - 6 + 4 = 2$

4. 

   cube with a hole

   $\chi = 16 - 32 + 16 = 0$

5. 

   triangular prism with a hole

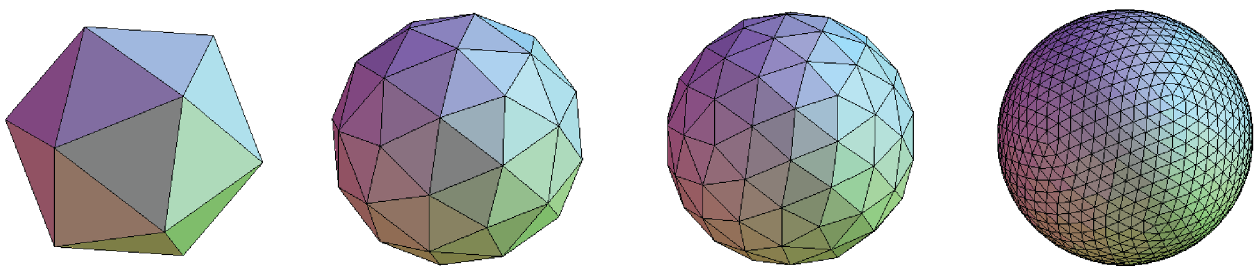
   $\chi = 12 - 24 + 12 = 0$

Look at the number of holes each surface has and its Euler characteristic. Do you notice any patterns?

It turns out, all homeomorphic surfaces have the same Euler characteristic, so we can deform a shape as much as we would like without cutting it, and the Euler characteristic will stay the same. Properties like this, which remain unchanged after applying certain types of transformations, are called **invariants**.

Did you notice the way in which we divided up the sphere into triangles? This is called **triangulation**.

Based on what we discovered, what is the Euler characteristic of the following four shapes?
2



Images by Theon, retrieved from `https://en.wikipedia.org/wiki/User:Tomruen/Geodestic_sphere`

Computer modelling softwares represent curved-surface models with triangulations. This is how movies are animated and 3D models are built. As the triangles get smaller and smaller, they look more and more like a smooth curved surface! Why do you think computers prefer triangulations to real curved surfaces?

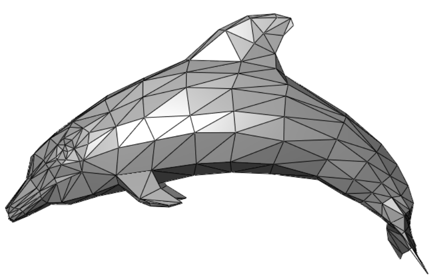
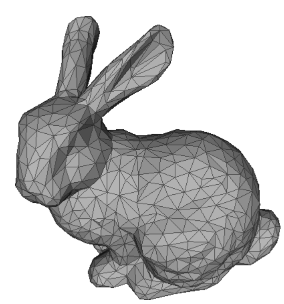


Image by Chrschn, retrieved from `https://commons.wikimedia.org/wiki/File:Dolphin_triangle_mesh.png`



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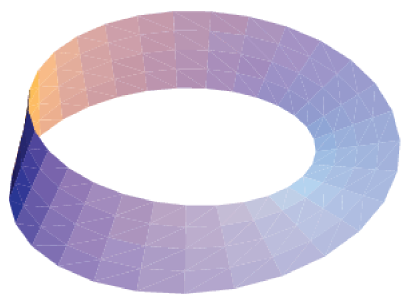
Möbius Strips

If I give you a strip of paper and some tape, what can you do to it so that you can trace your finger along both sides of it without ever crossing an edge?

The solution to this puzzle is a very important shape in topology, called the **Möbius strip**. It is a strange surface: it is 3D but only has 1 side! To create it, connect the two ends of your paper as if you were going to make a loop, but twist one end 180° before gluing it to the other end.



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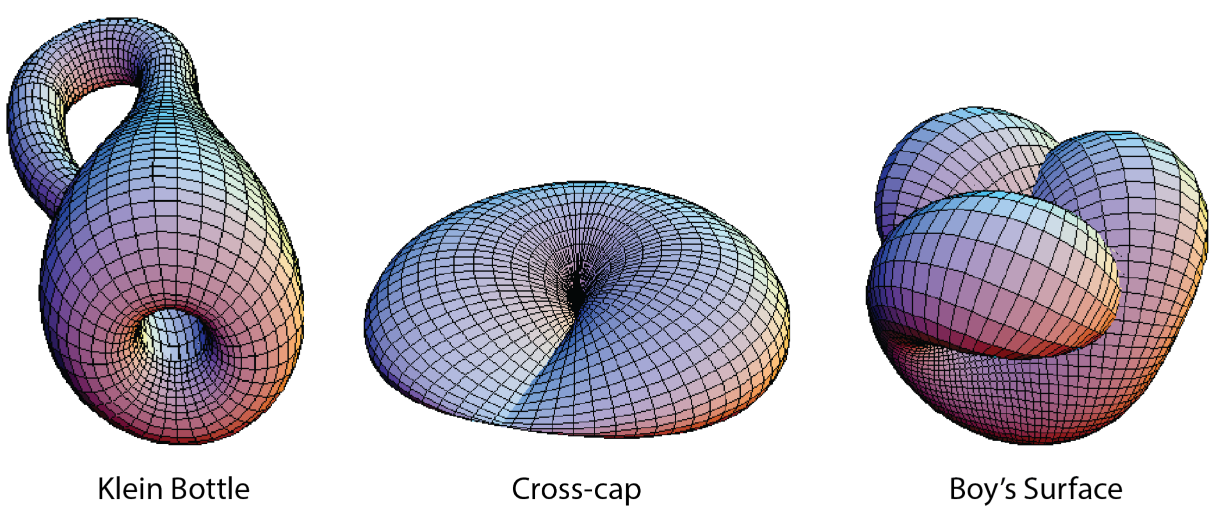
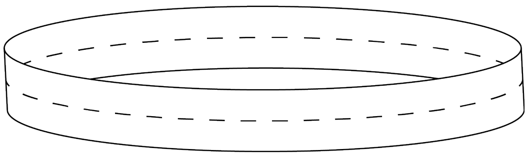
The Möbius strip does not have a distinguishable "inside" and "outside" like other 3D shapes. There is no way to consistently define an "orientation" everywhere on the strip, so it is called a **non-orientable** surface. The following are 3D approximations of some other non-orientable surfaces (we cannot really represent these without having 4 dimensions):



Images by AugPi, retrieved from `https://commons.wikimedia.org`

Let's say I have two identical strips of paper. They are each 1 cm thick and 10 cm long. I turn one strip into a loop (0 twists) and the other into a Möbius strip (1 twist).

If I cut along the $\frac{1}{2}$ point of the loop's width, what will I get?

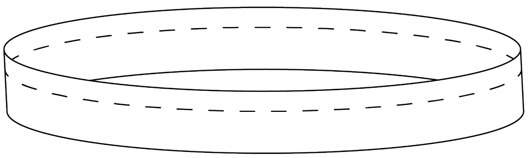


You will end up with two loops, each 0.5 cm thick and 10 cm long.

What do you think will happen if I do the same with the Möbius strip?

Surprisingly, you will end up with one long loop with *two* twists. It will be 0.5 cm thick and 20 cm long.

What if I cut along the $\frac{1}{3}$ point of the loop's width?



You will end up with two loops, both 10 cm long, one $\frac{1}{3}$ cm thick and the other $\frac{2}{3}$ cm thick.

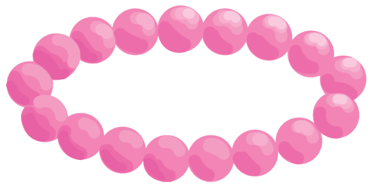
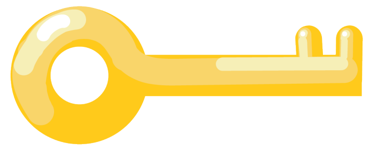
What if I do the same with the Möbius strip?

Another surprising result! You will end up with a 10 cm long Möbius strip linked to a 20 cm long loop with two twists. They will both be $\frac{1}{3}$ cm thick.

Problem Set

Problems marked with an asterisk (*) are challenge questions.

1. Calculate the genus of the following objects:

(a)  (b)  (c)  (d) 

0 1 2 1

2. Which of the objects above are homeomorphic?

   The bracelet and the key.

3. Come up with two homeomorphic objects of your own.

   Answers may vary. One example is an ice cube and a rubber ball.

4. (a) What is the genus of a paper loop?

   1

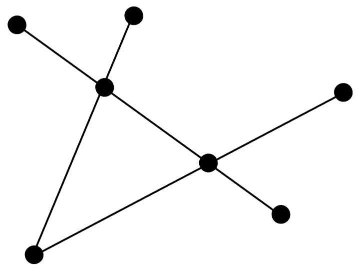
   (b) What is the genus of a Möbius strip?

   1

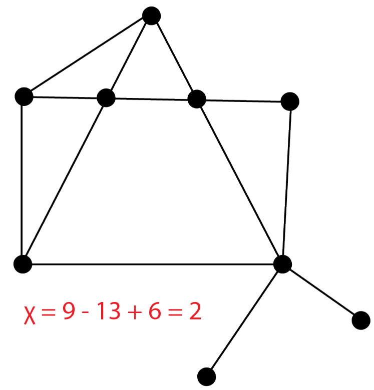
   (c) * Are a paper loop and a Möbius strip homeomorphic? *(hint: what would you have to do to transform one into the other?)*

   No. Recall that the definition of homeomorphic is that one object can be transformed into the other by stretching and deforming, but not by cutting! The only way to make a Möbius strip from a regular loop is by cutting the loop, twisting it, and reattaching it, so they are not homeomorphic.

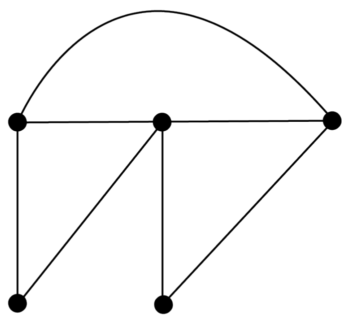
5. Calculate the Euler characteristic of the following graphs:



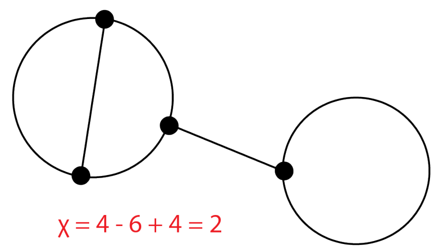
χ = 7 - 7 + 2 = 2



χ = 9 - 13 + 6 = 2

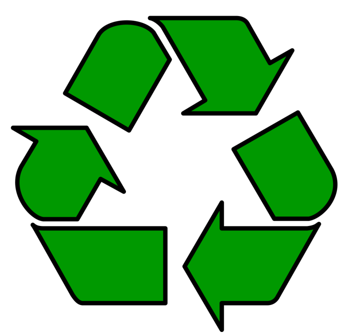


χ = 5 - 7 + 4 = 2



χ = 4 - 6 + 4 = 2

6. This is the universal recycling symbol. Do you notice anything interesting about it?



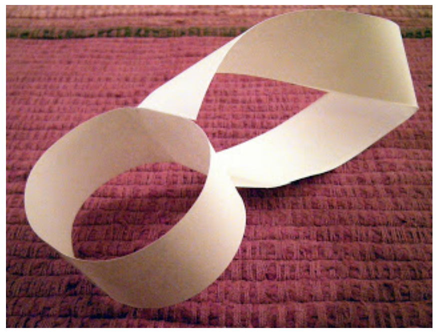
Retrieved from `https://commons.wikimedia.org/wiki/File:Recycle001.svg`

It's a Möbius loop!

7. To make a Möbius strip, we took a strip of paper and twisted it (180°) once before gluing the ends together. Make a strip with two twists, and cut along the halfway point of the width like we did with the Möbius strip. What happens? Try it with three twists, or four twists. Do you see a pattern?

There are a number of patterns to be observed, especially regarding the number of twists in the result. The most basic pattern is that an odd number of twists will result in a single strip, and an even number of twists will result in two linked strips.

8. The following exercise was invented by Martin Gardner. Make a prediction and then try it yourself: Make a cross out of paper. Fold one set of arms into a loop, and the other into a Möbius band like so:



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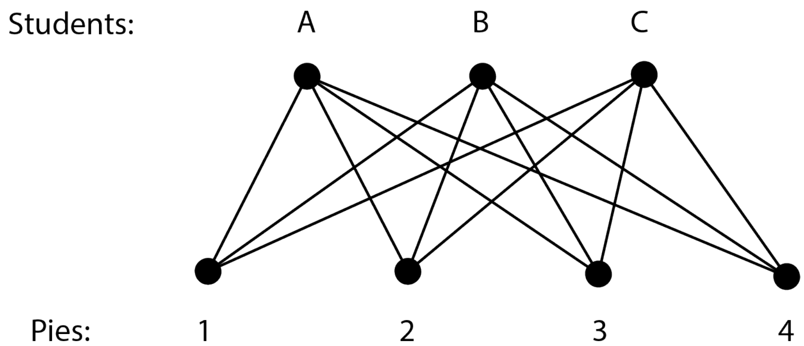
`http://mathtourist.blogspot.com/2010/10/martin-gardners-mobius-surprise.html`

Cut the Möbius band along the $\frac{1}{3}$ point of its width. Then cut the loop along the $\frac{1}{2}$ point of its width. What happens?

Surprisingly, you get a Möbius loop linked to a square!

Extension: Graphs, which we looked during this lesson, can be used to simplify certain problems. We can represent objects as vertices and connections between these objects as edges between the vertices.

For example, suppose we have 3 students and 4 pies. If each student wants to take one piece from each pie, how many pieces of pie will be eaten? We can represent this problem using a graph, with each student or pie as a vertex, and edges to represent a student taking a piece of pie.



Counting up the number of edges, we can see that 12 pieces of pie were eaten.

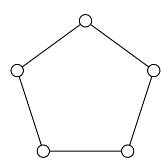
The **degree** of a vertex is the number of vertices it is connected to. One useful theorem for solving graph problems is that if the sum of the degrees of all the vertices of a graph is D, and the number of edges is E, then

$$D = 2 \times E$$

The remaining questions in the problem set will involve graph theory.

9. (a) * There are 5 people at a party. Is it possible for each of them to know exactly 2 people?

   This is possible. The following graph shows 5 people (vertices), each with degree 2.

   (b) * Is it possible for each of them to know exactly 3 people?

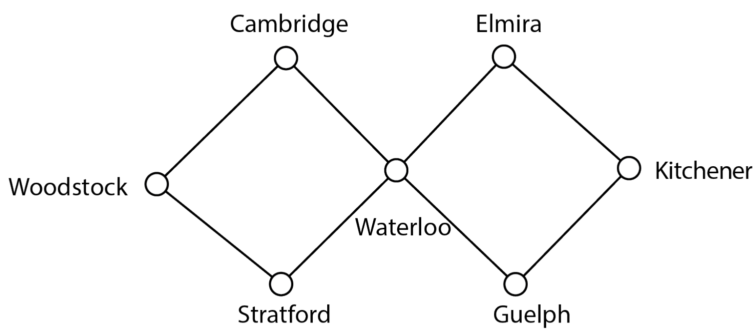
   After trying to draw a graph with 5 vertices, each with degree 3, you might think that this is not possible. You are right! To prove it, we can use the formula:

$$\begin{aligned} D &= 2 \times E \\ 3+3+3+3+3 &= 2 \times E \\ 15 &= 2 \times E \\ 7.5 &= E \end{aligned}$$

   But E is the number of edges. We cannot have half an edge, so this is not possible.

10. * Yara is a cyclist who lives in Waterloo. She wants to start in Waterloo, bike through all of the cities on the following map exactly once, and then return to Waterloo. The map below shows the bike roads available to her. Is it possible?

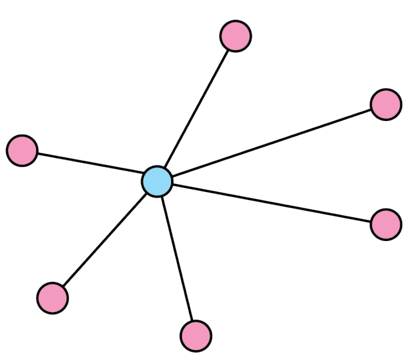
    It is not possible. The map below shows how to "untangle" the original graph to reveal that it is composed of two loops. It is impossible to move from the first loop to the second loop without passing through Waterloo an extra time, so it is not possible for Yara to make the type of trip she would like.

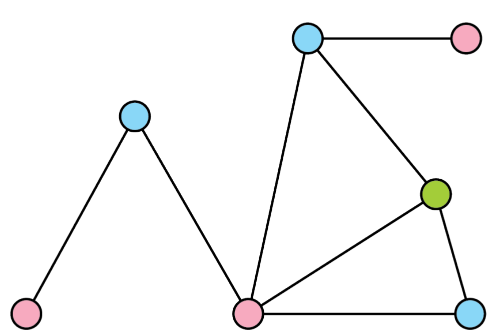


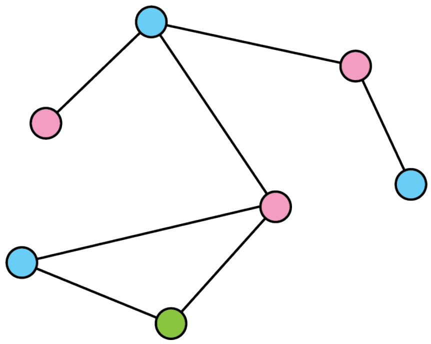
11. * David, Yara's American friend, is also a cyclist. Below is a map of the bike paths available to him. If he wants to start and end his trip in Downtown Portland, and visit each district exactly one time, what route should he take?

    The following path satisfies the requirements: Downtown Portland → Montavilla → Richmond → Woodlawn → Irvington → Kenton → Downtown Portland

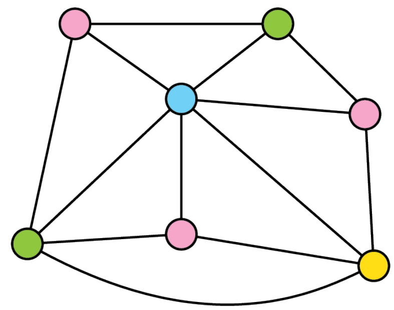
12. * The **chromatic number** of a graph is the number of colours required to colour each vertex such that any vertices connected by an edge are different colours. Find the chromatic number of each graph below:

(a)  2

(b)  3



(c) 3

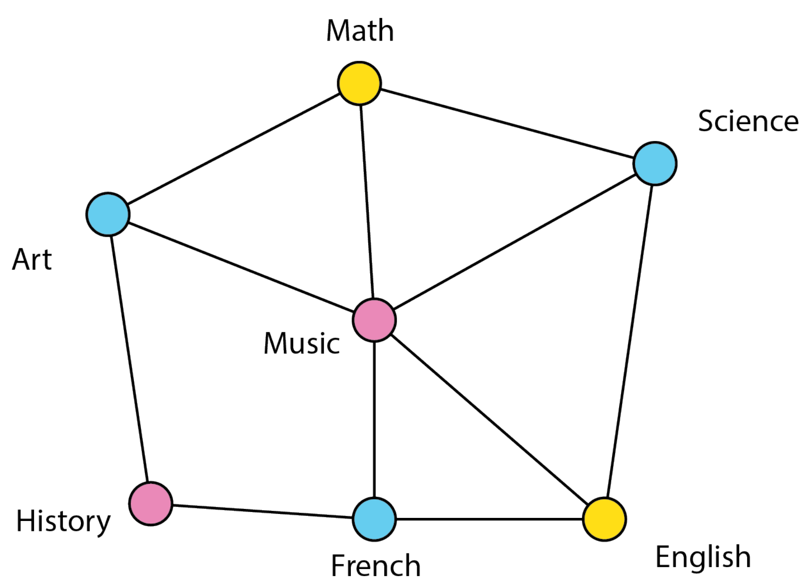


(d) 4

13. * Graph colouring (and finding the chromatic number) has a lot of applications. Try to use graph colouring to solve the following problems:

    (a) Suppose there are 7 subjects offered at a school for grade 6 students. Each teacher wants to have an exam during lunch time. However, there are students taking multiple subjects, so classes that share students have to have exams on different days. What is the minimum number of days required to run all 7 exams?

        The following graph represents the problem. Each vertex is a subject, each edge represents shared students. A colouring of this graph represents the different days the exams can be held on. The chromatic number of this graph is 3, so all 7 exams can be run in 3 days.



(b) Most maps use different colours for neighbouring regions. Below is a map of Africa. What is the least number of colours required to colour it in so that countries which share a border are different colours?

Only 4 colours are required!

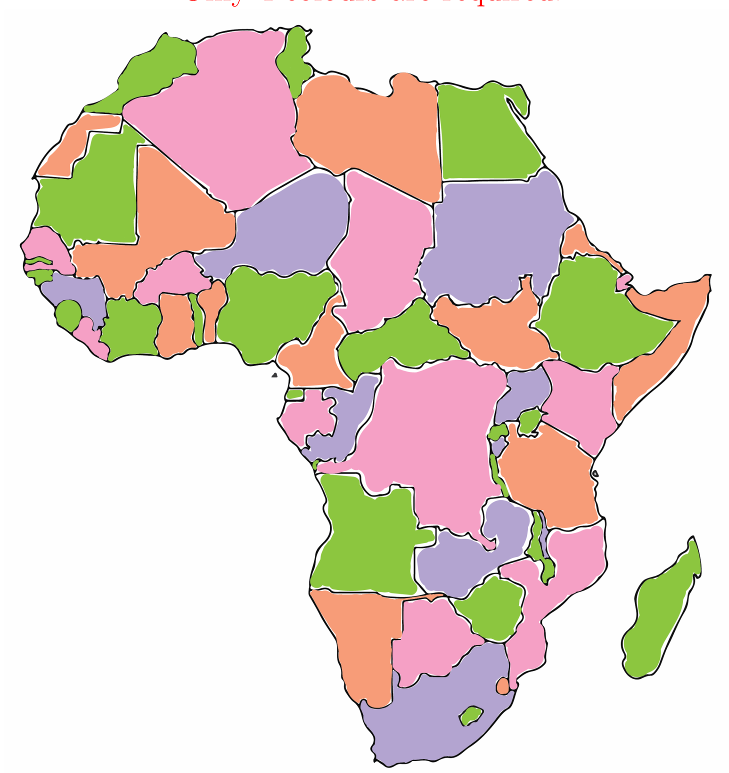


Image by Bruce Jones, retrieved from `http://www.freeusandworldmaps.com`

There is a famous theorem called the **four color theorem** that states that any map can be coloured in this way using at most 4 colours.