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Grade 6 Math Circles

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Counting Part I - Solutions

Today we will be learning how to count the number of possible outcomes for specific actions and events. This type of mathematical counting is the foundation of an entire branch of math called **combinatorics**. How is this different from counting "1, 2, 3,. . . "? Let's find out!

How Many Choices?

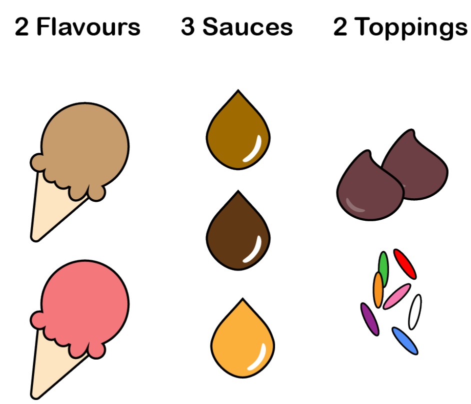
Example: Ice Cream, Ice Cream!

Suppose you are making an ice cream sundae. There are 2 flavours, 3 sauces, and 2 toppings that you can choose from. You are to select one of each. How many different sundaes could you make?

Ice Cream, Ice Cream! Video Solution:
https://youtu.be/8Y8GbQ0lpZc

With 2 flavours, 3 sauces, and 3 toppings you can make 12 different sundaes.

What if I had 8 flavours, 6 sauces, and 4 toppings? We can solve this by writing down each option like we did in the video above. You may notice that with more options, using this method will take a very long time! How can we calculate this more effeciently?



Fundamental Counting Principle:
If you have to make Choice A **AND** Choice B, and there are m options for Choice A and n options for Choice B, then the total number of different ways you can make Choice A and Choice B is $m \times n$.

This principle applies in the same way if more choices have to made. We simply have to multiply the options for each choice.

When you see "**AND**" in a counting problem, it is a hint that you have to use this principle! Let's apply this rule to the Ice Cream, Ice Cream! example.

Example: Ice Cream, Ice Cream! continued

First let's count the total number of choices and options that we have.

- Since we can choose an ice cream flavour, a sauce, and a topping for each sundae, we have **3** total choices.
- For options, there are **2** options for flavour, **3** options for sauce, and **2** options for topping. Now using the Fundamental Couting Principle we have:

$$\text{total number of different sundaes} = \underset{\text{flavours}}{2} \times \underset{\text{sauces}}{3} \times \underset{\text{toppings}}{2} = 12$$

This is the same answer that we got using the earlier method!

We can also use this approach to solve the total number of different sundaes given 8 flavours, 6 sauces, and 4 toppings. We still have **3** total choices. For options, there are **8** options for flavour, **6** options for sauce, and **4** options for topping. Now using the Fundamental Couting Principle we have:

$$\text{total number of different sundaes} = \underset{\text{flavours}}{8} \times \underset{\text{sauces}}{6} \times \underset{\text{toppings}}{4} = 192$$

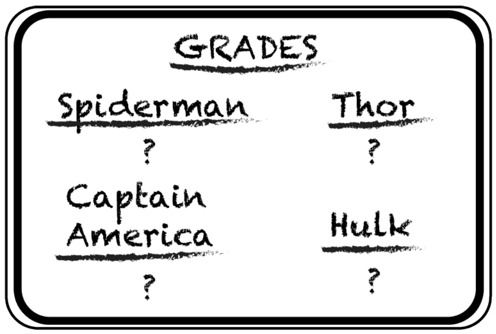
Imagine having to count 192 different sundaes like we did in the video. The Fundamental Couting Principle made counting this much simpler!

Repetition

In this lesson, we will look at two types of counting problems where **repetition is allowed**, and **repetititon is not allowed**. What does "repetition" mean in this case? Let's take a look at some examples.

Example: The Best Hero

Iron Man gives Spiderman, Thor, Captain America, and Hulk a task each to test their fighting abilites. Each hero could get one of 6 grades: **Outstanding, Exceeds Expectations, Acceptable, Poor, Dreadful, or Atrocious.** Iron man will then call a meeting with the Avengers and write their grades on the board. Assume it doesn't matter which heroes' grade he lists first, second, third, or fourth on the board.



1. Can Iron Man repeat any grades (i.e. give the same grade to more than one hero)?

2. How many different ways can the heroes be graded? Use Fundamental Counting Principle (FCP).

Solution:

1. Yes! - Since the same grade can be given to more than 1 hero, he can repeat grades. This is an example of a case where **repetition is allowed**. When repeptiton is not allowed the number of options for each choice is the same.

2. Using the FCP there are 4 choices Iron Man needs to make. He has to choose each of the **4 heroes' grade**. There are 6 options for each grade, **Outstanding, Exceeds Expectations, Acceptable, Poor, Dreadful, or Atrocious**. Thus:

$$\text{the total number of different ways to grade} = 6 \times 6 \times 6 \times 6 = 1296$$

Example: Bake a Cake

Suppose you are making a four-layer cake and have four flavours of cake to choose from: vanilla, chocolate, strawberry, and red velvet. Each layer of this cake must be a different flavour.

1. Is repetition allowed (i.e. using the same flavour more than once)?
2. How many different four-layer cakes can you make?

Bake a Cake Video Solution: *https://youtu.be/A3-BFZu-KFk*

1. No repetition is not allowed! This is because the question specifies that each layer of the cake must be a different flavor.

2. Using the FCP there are 4 choices we need to make. We need to choose a flavor for the first layer, the second layer, the third layer, **and** the fourth layer of our cake. We need to calculate:

   (Options for layer 1)×(Options for layer 2)×(Options for layer 3)
   ×(Options for layer 4)

   There are 4 options for the first layer (since all 4 flavours are still unused). Then, our equation becomes: 4×(Options for layer 2)×(Options for layer 3)
   ×(Options for layer 4)

   Now 1 of the flavours has now been used (in the first layer), so for layer 2 there are 3 options. Our equation becomes: $4 \times 3\times$(Options for layer 3)×(Options for layer 4)

   Now two flavours will have already been used (in the first and second layer) so there are 2 options for the third layer. We have: $4 \times 3 \times 2\times$(Options for layer 4)

   Lastly, three flavours have been used (in the first, second, and third layer) so there is only one option left for the fourth layer. Then:

   The total number of diffferent four layer cakes $= 4 \times 3 \times 2 \times 1 = 24$

In this example **repetition was not allowed**. When repetition is not allowed we have to reduce the number of available options each time.

Factorials

Whenever we have a list or group of n things and we need to figure out how many different ways they can be put in with **no repetition**, we use the same type of logic that we just applied to the previous example. This will always result in the answer being $n \times (n-1) \times (n-2) \times (n-3) \times ... \times 2 \times 1$. In our previous example our answer was $4 \times 3 \times 2 \times 1$. In other words, you are using the Fundamental Counting Principle to multiply the options for each choice in the order starting at n and decreasing by 1 every time a choice in the order is filled.

Factorial notation is used to write this operation:

$$n! = n \times (n-1) \times (n-2) \times ... \times 2 \times 1$$

$n!$ is read as "n-factorial".

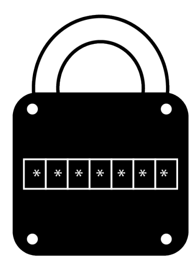
Examples:

1. $0!$ is a special case. We can say that $0! = 1$
2. $1! = 1$
3. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
4. $11! = 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 39916800$

Example: What's the Password?

Harry and Ron are trying to figure out the password for a secret passage behind the mirror on the fourth floor. They know the 7-digit long password uses the whole numbers from 1 to 7 but they do not know what order.

1. How many different passwords could there be if the digits **can repeat**?
2. How many different passwords could there be if the digits **cannot repeat**?



Solution:

1. There are 7 **choices** since the password is 7-digits long. Since repetition is allowed (i.e. we **can repeat** digits), there are 7 **options** for each digit (the numbers 1 to 7). Then,

$$\begin{aligned}\text{total number of different passwords} &= \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{7} \times \underset{\text{digit 3}}{7} \times \underset{\text{digit 4}}{7} \times \underset{\text{digit 5}}{7} \times \underset{\text{digit 6}}{7} \\ &\times \underset{\text{digit 7}}{7} \\ &= 823,543\end{aligned}$$

2. Since repetition is not allowed (i.e. the digits **cannot repeat**) we can only use each number once. For the first digit of our password we have 7 **options** to choose from. The digits: 1, 2, 3, 4, 5, 6, or 7.

$$\text{total number of different passwords} = \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{} \times \underset{\text{digit 3}}{} \times \underset{\text{digit 4}}{} \times \underset{\text{digit 5}}{} \times \underset{\text{digit 6}}{} \times \underset{\text{digit 7}}{}$$

For the second digit of our password we have 6 **options** to choose from, since we chose a number from 1 to 7 for our first digit.

$$\text{total number of different passwords} = \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{6} \times \underset{\text{digit 3}}{} \times \underset{\text{digit 4}}{} \times \underset{\text{digit 5}}{} \times \underset{\text{digit 6}}{} \times \underset{\text{digit 7}}{}$$

For the third digit of our password we have 5 **options** to choose from, since we chose 2 numbers from 1 to 7 for our first and second digit.

$$\text{total number of different passwords} = \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{6} \times \underset{\text{digit 3}}{5} \times \underset{\text{digit 4}}{} \times \underset{\text{digit 5}}{} \times \underset{\text{digit 6}}{} \times \underset{\text{digit 7}}{}$$

We notice that this is just the factorial of 7 since there is **no repeptition**. Thus,

$$\begin{aligned}\text{total number of different passwords} &= 7! \\ &= \underset{\text{digit 1}}{7} \times \underset{\text{digit 2}}{6} \times \underset{\text{digit 3}}{5} \times \underset{\text{digit 4}}{4} \times \underset{\text{digit 5}}{3} \times \underset{\text{digit 6}}{2} \\ &\times \underset{\text{digit 7}}{1} \\ &= 5040\end{aligned}$$

Permutations

What if we had n options in total to choose from but we only needed to order k of them? For example, suppose we wanted to know how many different 4-digit long passwords there are using the whole numbers from 1 to 7, without repeating digits. We have 7 (n) options but only need to order 4 (k) of them.

> **Permutations** are a way of counting in this type of situation when there is **no repetition**.
>
> $$_nP_k = \frac{n!}{(n-k)!}$$
>
> This is read as "n permute k" and counts how many ways we can order k objects from a total of n objects

Example: Quidditch Tryouts

The National Quidditch team is holding tryouts for one Keeper, one Seeker, and one Chaser. Dimitar, Viktor, Georgi, Boris, Bogomil, Nikola, and Stoyanka all try out.

1. How many players total do we have to choose from and how many do we need to choose?
2. Is repetition allowed in our choices? Why?
3. Use the Fundamental Counting Principle to find how many ways the three positions can be filled.
4. Now use the permutation formula to find how many ways the three positions can be filled.

Quidditch Tryouts Video Solution: *https://youtu.be/XUaake50tz0*

1. There are 7 players to choose from in total and we need to choose 3.
2. No, there is no repetition allowed because once a player is picked for one position, they can't also be picked for another postition.

3.

$$\begin{aligned}
\text{number of ways} &= (\text{options for Keeper}) \times (\text{options for Seeker}) \times (\text{options for Chaser}) \\
&= 7 \times 6 \times 5 \\
&= \frac{7!}{4 \times 3 \times 2 \times 1} \\
&= \frac{7!}{4!} \\
&= 210
\end{aligned}$$

4. We already figured out that $n = 7$ and $k = 3$.

$$\text{number of ways} = {}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \times 6 \times 5 \times 4!}{4!} = 7 \times 6 \times 5 = 210$$

Example: How many different 3-letter words can you make using the letters from the word FLOWER with no repetition (i.e. you can only use each letter once in the 3-letter word)? They don't have to be valid words in English.

Solution:

In this example, **repetition is not allowed** since we can use each letter only once. This means we can use the permutation formula to find the number of different 3-letter words that we can make. Here $n = 6$ since we have 6 letters in the word FLOWER, and $k = 3$ since we need to choose/order 3-letters. Then,

$${}_nP_k = \frac{n!}{(n-k)!} = {}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 6 \times 5 \times 4 = 120$$

Using the word FLOWER we can make 120 different 3-letter words.

Example: How many different 7-character licence plates can be made if the first 4 characters must be digits from 0 to 9 with no repetition, and the last 3 characters must be letters of the alphabet with repetition? (**Hint:** Remember the Fundamental Counting Principle)

Solution:

- For the first 4 characters since **repetition is not allowed**, using digits from 0-9, there are **10** options for the first digit, **9** options for the second digit, **8** options for the third digit, and **7** options for the fourth digit.
- For the last 3 characters, we have **26** options (A to Z) for each letter since **repetition is allowed.** Thus,

$$\begin{aligned}\text{total number of different license plates} &= \underset{\text{digit 1}}{10} \times \underset{\text{digit 2}}{9} \times \underset{\text{digit 3}}{8} \times \underset{\text{digit 4}}{7} \times \underset{\text{letter 1}}{26} \times \underset{\text{letter 2}}{26} \\ &\quad \times \underset{\text{letter 3}}{26} \\ &= 88583040\end{aligned}$$

Problem Set

1. Spiderman decided that he wants to fight today. He needs to pick a villain, a location, and a teammate. Here's are Spiderman's options:

   - **Villain**: Scorpion, Carnage, or Mysterio
   - **Location**: The Bronx, Queens, Brooklyn, or Manhattan
   - **Teammate**: Iron Man, Thor, Captain America, Black Widow, Hulk, or Black Panther

   How many different fights could Spiderman choose from?

   Using the Fundamental Counting Principle we have:

   total number of different fights $= 3 \times 4 \times 6 = 72$

2. Plankton is only a few steps away in getting his hands on the Krabby Patty secret formula. He needs to open a vault that has a 5-digit numerical password. How many different password combinations could he try?

   Since digits can repeat and there are 10 options for each digit (0 to 9), by the Fundamental Counting Principle:

   total number of diffferent passwords $= 10^5 = 100000$

3. A simplified version of a Canadian postal code is made up of 6 characters of the format "A1A 1A1" where A is a letter and 1 is a digit (from 0 to 9). How many different postal codes could there be?

   There are 26 letters in the alphabet and 10 digits (from 0 to 9). Repetition is also allowed because the letters or numbers can repeat. So, according to Fundamental Counting Principle:

   total # of different postal codes $= 26 \times 10 \times 26 \times 10 \times 26 \times 10 = 26^3 \times 10^3 = 17,576,000$

4. Solve these factorial problems:

   (a) $8! = 40320$

   (b) $\dfrac{6!}{3!} = \dfrac{6 \times 5 \times 4 \times 3!}{3!} = 6 \times 5 \times 4 = 120$

   (c) $\dfrac{17!}{14!3!2!} = \dfrac{17 \times 16 \times 15 \times 14!}{14! \times (3 \times 2 \times 1) \times (2 \times 1)} = \dfrac{17 \times 16 \times 15}{3 \times 4} = 17 \times 4 \times 5 = 340$

(d) ${}_7P_3 = \dfrac{7!}{(7-3)!} = \dfrac{7!}{4!} = 210$

5. Write these expressions using factorials:

   (a) $1 = 1! = 0! = \dfrac{n!}{n!}$

   (b) $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$

   (c) $100 \times 99 \times 98 \times 97 = \dfrac{100!}{96!}$

   (d) $15 \times 14 \times 3 \times 2 \times 1 = \dfrac{15!}{13!} \times 3! = \dfrac{15!3!}{13!}$

6. The NBA has to schedule basketball matches between 4 different teams. The Toronto Raptors, the Boston Celtics, the Los Angeles Lakers, and Miami Heat. Each team plays each of the other teams once.

   (a) How many matches in total are there?

$$3 + 2 + 1 = 6 \text{ matches.}$$

   The Raptors have to play the 3 other teams, the Boston Celtics, the Los Angeles Lakers, and Miami Heat. Then the Boston Celtics have to play the 3 other teams but they've already played the Raptors so that match is already counted. Then the Los Angeles Lakers have to play the 3 other teams but they've already played the Raptors and the Boston Celtics so they only have to play Miami Heat. And now, Miami Heat has already played all other teams.

   (b) How many different ways are there to schedule the matches?

   We need to arrange 6 games with no repeats. Here order matters, thus:

$$\text{total number of different schedules} = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6! = 720$$

7. Spongebob, Patrick, Mr. Krabs, Squidward, Plankton, Sandy, and Gary, are having a race in their Boatmobiles. There is a prize of have a small, a medium, and a large Magic Conch. The first, second, and third place winners will get the large, medium, and small magic conch respectively. How many ways can the Magic Conches be given out?

   There is no repetition because someone cannot finish the race twice. So, we use permutation. There are 7 competitors in total and 3 Magic Conches:

$${}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 7 \times 6 \times 5 = 210$$

8. (a) How many ways can the letters of the word GUITAR be arranged?

Since there are no repeated letters in the word GUITAR, we have 6 options for the first letter, 5 options for the second letter, 4 options for the third letter, and so on. Thus:

$$\text{total number of arrangments} = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

(b) How many 3-letter words can you make using letters of the word GUITAR?

Since there are 6 letters and we want to order 3 of them, with no repetiton, we can use permutation:

$${}_6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 6 \times 5 \times 4 = 120$$

(c) How many 3-letter words can you make using letters of the word GUITAR?

Since G needs to be in the first spot, there is only one option for the first letter. For the second letter, there are 5 options since the letter G has been used for the first letter. Then there are 4 options for the third letter, 3 options for the fourth letter, 2 options for the fifth letter, and one option for the sixth letter.

total number of arrangements $= 1 \times 5 \times 4 \times 3 \times 2 \times 1 = 120$

9. (a) How many 7-digit passwords are there with no repeating digits?

We need to order 7-digits with no repeats. There are 10 options for the first digit (0 to 9). Then 9 options for the second digit, 8 options for the third digit, and so on. Thus:

$$\text{total number of different passwords} = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 7! = 604800$$

(b) How many 6-digit numbers are there with no repeating digits? (**Hint:** Can a 6-digit number start with a 0?)

We need to order 6-digits with no repeats. Here the first digit cannot be a 0 or else that will be a 5-digit number. Thus the first digit only has 9 options since we are not including 0. The second digit will also have 9 options since we can now include 0. Thus:

$$\text{total number of different passwords} = 9 \times 9 \times 8 \times 7 \times 6 \times 5 = 136080$$

(c) How many 6-digit **even** numbers are there with repeating digits? (**Hint:** Use the Fundamental Counting Principle)

Since the number must be even, we know that the last digit must be even so there are 5 options. We can choose 0, 2, 4, 6, or 8. The first digit only has 9 options (0 cannot be used in the first position). The second, third, fourth, and fifth number can be any of the 10 digits. Thus using the Fundamental Counting Principle:

$$\text{total number of different passwords} = 9 \times 10 \times 10 \times 10 \times 10 \times 5 = 45000$$

10. **Challenge problem:** Luke, Princess Leia, Han Solo, Chewbacca, C-3PO, Yoda, and R2D2 are taking a ride on the Millennium Falcon. How many ways ways can they sit if R2D2 and C-3PO must sit together and Han Solo cannot sit beside Luke? Assume the seats are arranged in a line.

    Let's break down this problem into each condition. The first condition says R2D2 and C-3PO must sit together. Since R2D2 and C-3PO need to stay together, we can think of them as one person and then multiply by the number of ways both of them can be arranged. You can picture R2D2 and C-3PO as the same person and wherever they are sitting, we know they can be arranged in 2 different ways, this is simply **R2D2-C-3PO** and **C-3PO-R2D2**. And so we multiply the total number of ways by 2. To get the total number of ways we count 6 people in total since we think of R2D2 and C-3PO as 1 person. We know ordering 6 people without repetition is simply 6!. Thus:

    $$\text{total number of ways} = 6! \times (2) = 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times (2) = 1440$$

    We also have the restriction that Han Solo cannot sit beside Luke. This is a little more difficult to imagine and so instead, let's find out the number of ways Han Solo **can** be beside Luke (while R2D2 and C-3PO are also beside each other) and then subtract it from the total number of ways found above. We will be left with the total number of ways they can sit if R2D2 and C-3PO must sit together **and** Han Solo cannot sit beside Luke. If we think of R2D2 and C-3PO as one person and also Han Solo and Luke as one person, then we count 5 people. Thus the number of ways both R2D2 and C-3PO are together **AND** Han Solo and Luke are together is:

    $$5! \times (2) \times (2) = 5 \times 4 \times 3 \times 2 \times 1 \times (2) \times (2) = 480$$

    Then, our final answer is:

    $$(6! \times 2) - (5! \times 2 \times 2) = 1440 - 480 = 960$$

11. **Challenge problem:** How many different 10-letter "words" can you make using the letters in the word:

STATISTICS

Note: These do not have to be real words but do need to be made up of the above 10 letters arranged with no spaces or other symbols in between.

Hint: Think about the repeated letters in the word "STATISTICS".

There are 10 letters to order but the letters have repeats:

- 3 of S
- 3 of T
- 2 of I

To deal with these repeats, we divide by the number of ways these repeated letters can be arranged because they do not result in "different" words. (ie. if you order the word "AAB", you will only have $\frac{3!}{2!} = 3$ unique words which are "BAA", "ABA", and "AAB" rather than $3! = 6$ because it does not matter how you arrange the 2 A's.)

So, our answer is:

$$\text{total number of different words} = \frac{10!}{3!3!2!} = 50400$$