

Grade 7/8 Math Circles

October 21, 2020

Graph Theory Proofs - Solutions

Introduction

Graph theory is a field of mathematics that looks to study objects called graphs. The ideas and understanding gained from studying graphs can be applied to many other problems. Examples of these problems include matching organ donors to patients, finding the best routes from point A to B (like a GPS does) and even some of the problems that we looked at last week in the BCC prep. The key to the relation between problems and graphs is having rules that we know to be true about graphs, called theorems. This week we will look at different aspects of graphs and how we can show that certain things are true about them. In other words, how we can prove traits of certain graphs. First, we need to understand some basic properties of graphs.

Definitions

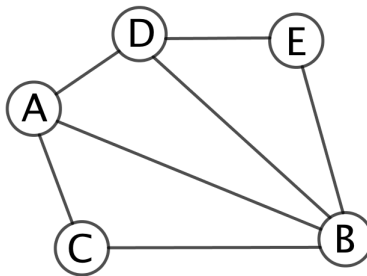


Figure 1: Graph used in examples below.

- A **vertex** (**plural: vertices**) is a point on a graph. Typically, it is represented by a small circle. It may also have a label.

Example: The circles labelled A , B , C , D and E in Figure 1 are vertices.

- An **edge** is a line that connects two vertices. The **endpoints** of an edge are the vertices that it connects.

Example: The segment connecting vertex A and vertex B is an edge. We typically write an edge, e , as $e = \{A, B\}$.

- A **graph** a graph is a collection of vertices with edges between some pairs of them.

Example: Figure 1 shows a graph.

- A **walk** is a sequence of vertices and edges that lead from one vertex to another. A **path** is a walk with no vertex repeated, except possibly the starting and ending vertex.

Example: $A, \{A, B\}, B, \{B, C\}, C, \{C, A\}, A, \{A, D\}, D$ is a walk in the example graph.

$A, \{A, B\}, B, \{B, C\}, C$ is a path.

- A **cycle** is a path that has the same starting and ending vertex. The length of a cycle is the number of distinct vertices in the path.

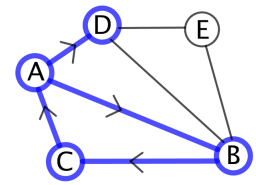
Example: $A, \{A, B\}, B, \{B, C\}, C, \{C, A\}, A$ represents a cycle of length 3 in the graph to the right.

- The **neighbours** of a vertex are all vertices that are connected to the vertex by an edge.

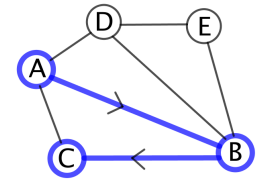
Example: The neighbours of vertex C are A and B .

- The **degree** of a vertex is the number of neighbours that it has.

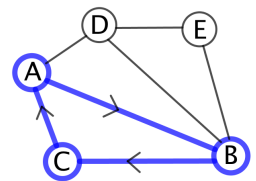
Example: Vertex C has degree 2.



Walk



Path



Cycle

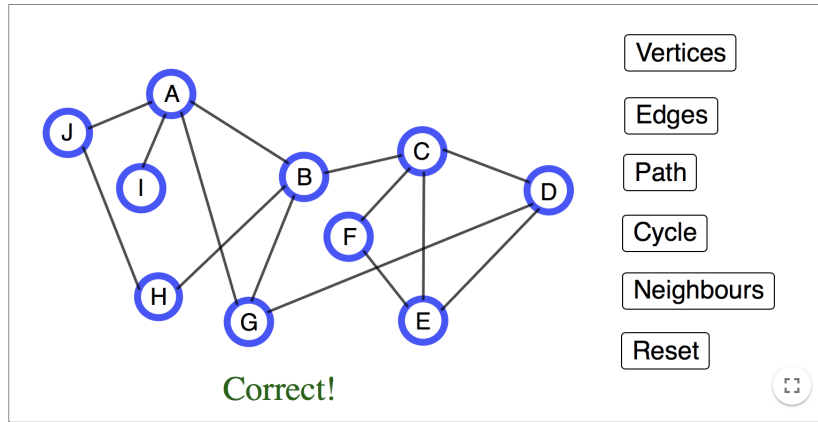
This week we will be looking only at **simple graphs**. A **simple graph** is a graph with three specific conditions.

1. Edges have no direction (we can travel in both directions on them when creating a walk or path).
2. There is most one edge between two vertices.
3. There are no loops. In other words, no edges that connect back to the same vertex.

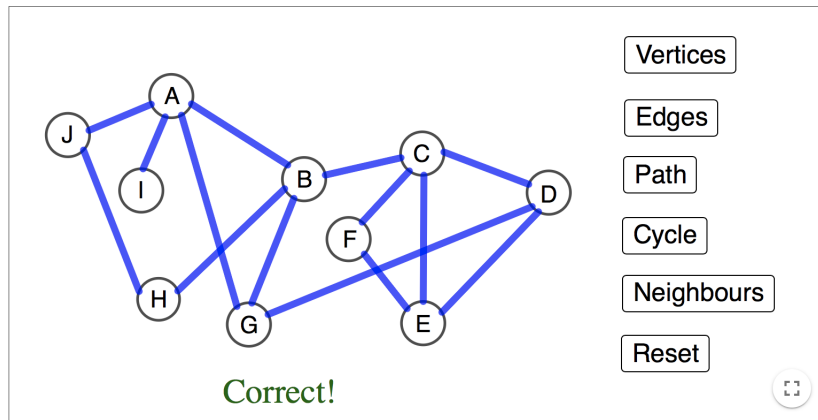
Exercise 1. Throughout this document there are online exercises to help practice what we have learned. Click on the link to try the activity: <https://www.geogebra.org/m/qaajsqya>

Solutions:

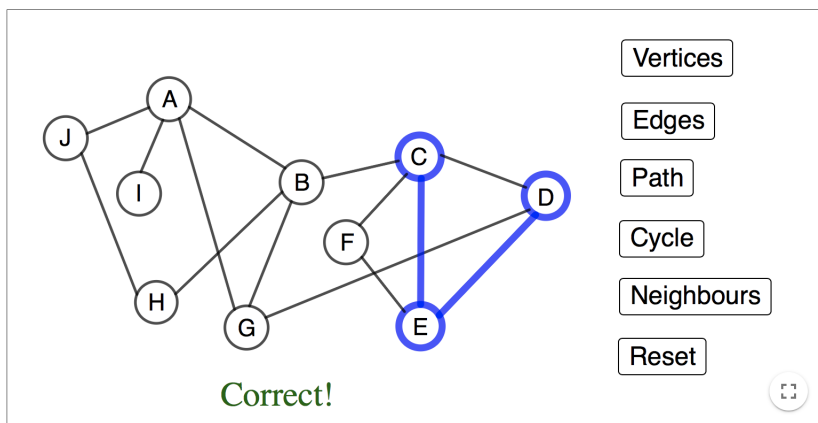
Vertices:



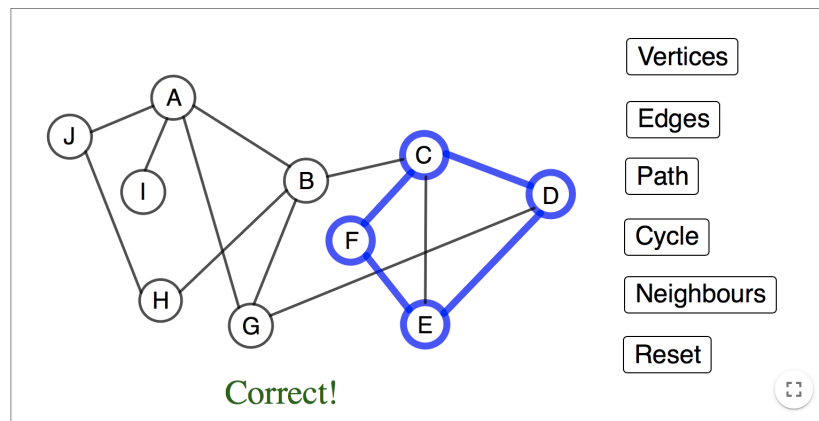
Edges:



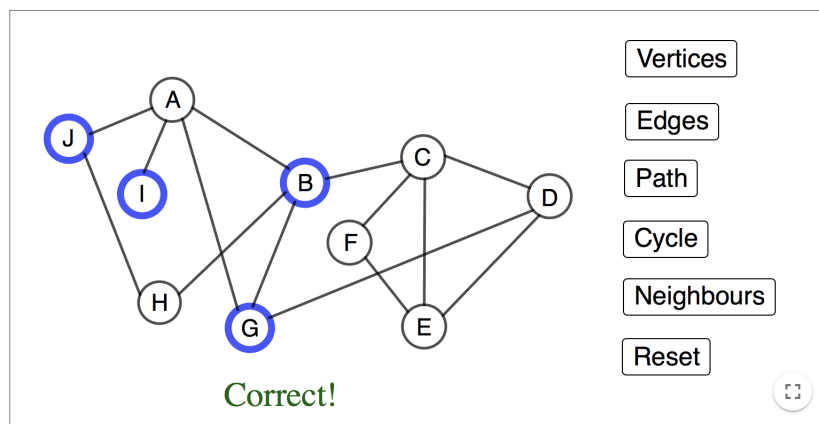
Path:



Cycle:



Neighbours:



In order to talk about graph theory proofs, we also need to understand what a proof is.

Definition 1. *Proof*

A mathematical proof shows conclusive evidence that a mathematical statement is true.

Once a statement has an accepted proof, it can be seen as a fact.

Planarity

Definition 2. *Planar embedding*

A planar embedding is a way to draw a graph where no edges and no vertices overlap.

Definition 3. *Planar Graph and Non-Planar Graph*

A planar graph is a graph that has a planar embedding. A non-planar graph has no planar embedding.

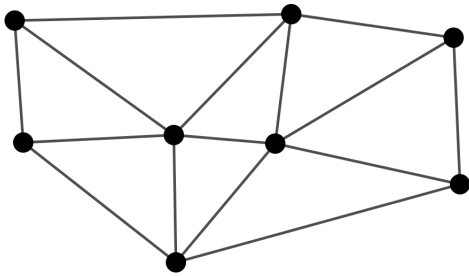


Figure 2: A planar embedding of a graph.

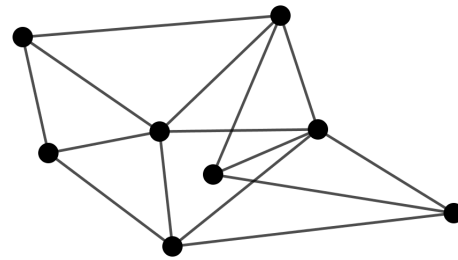


Figure 3: A non-planar embedding of the graph in Figure 2. Note that even if a graph is planar, not all of the ways that we can draw it will be a planar embedding.

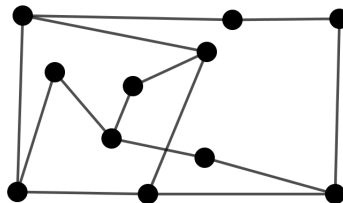
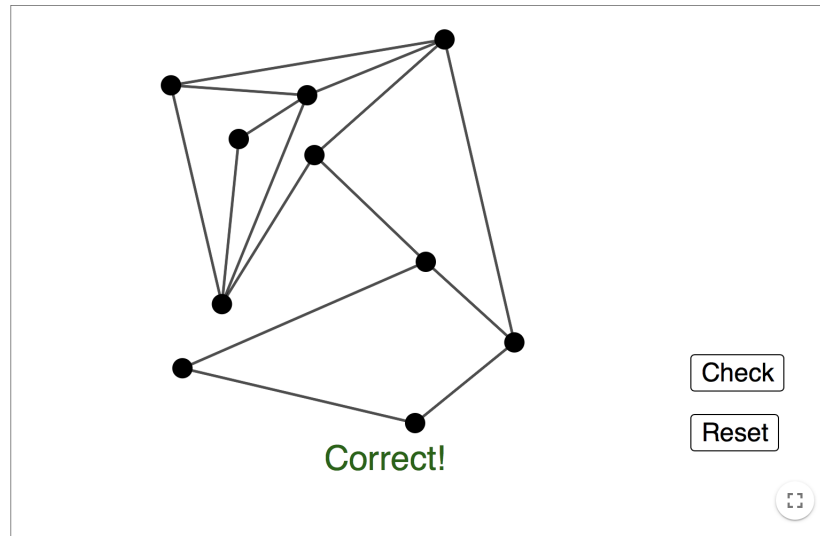


Figure 4: A nonplanar graph. There is no planar embedding of this graph. No matter how we arrange the vertices, we will always have edges that overlap. *Note: We will see this graph again in the Problem Set, where you can prove that it is indeed non-planar.*

Exercise 2. Find a planar embedding: <https://www.geogebra.org/m/rtyvz69a>

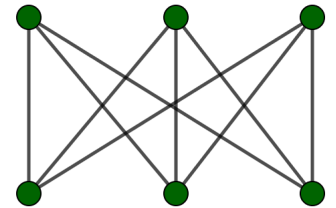
Solution:



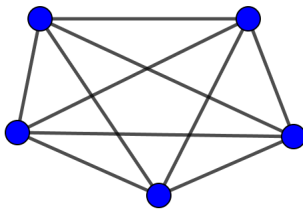
We can prove that a graph has a planar embedding (and so it is planar) by drawing a picture of the planar embedding. If we can show the planar embedding, then it must exist. But how do we prove that a graph is non-planar? There are a few theorems mathematicians use, but one definitive theorem is Kuratowski's Theorem. To understand Kuratowski's Theorem, we need a few more definitions.

Definition 4. $K_{3,3}$

The $K_{3,3}$ graph is a graph with 6 vertices, divided into 2 groups. Every vertex from each group is connected to every vertex in the other group. In the image given, the top three vertices are one group and the bottom three are another.



$K_{3,3}$



K_5

Definition 5. K_5

The K_5 graph is a graph with 5 vertices, where each vertex connects with each other vertex.

The $K_{3,3}$ and K_5 graphs are both non-planar. This online activity: <https://www.geogebra.org/m/pktcgcz6> will allow you to move them around to convince yourself of this. Note that this is not a proof, the proof that these are non-planar requires higher-level mathematics

that you will see in university. Rather than prove this fact, we're going to use it to prove other graphs are non-planar.

Definition 6. *Edge Subdivision*

We can create an edge subdivision of a graph by replacing each of its edges by a path. Watch this video: <https://youtu.be/iVFrR2ks9TA> to learn more.

We have now learned what we need to understand Kuratowski's Theorem.

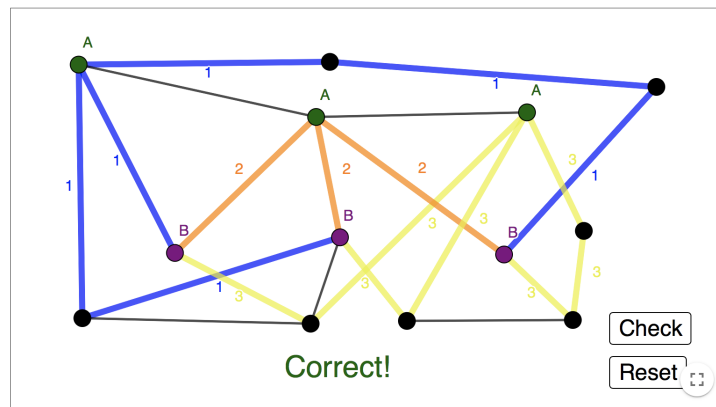
Theorem 1. *Kuratowski's Theorem*

A graph is planar if does not have an edge subdivision of either the $K_{3,3}$ or the K_5 graph as a part of the graph. If it does have such an edge subdivision, it is non-planar.

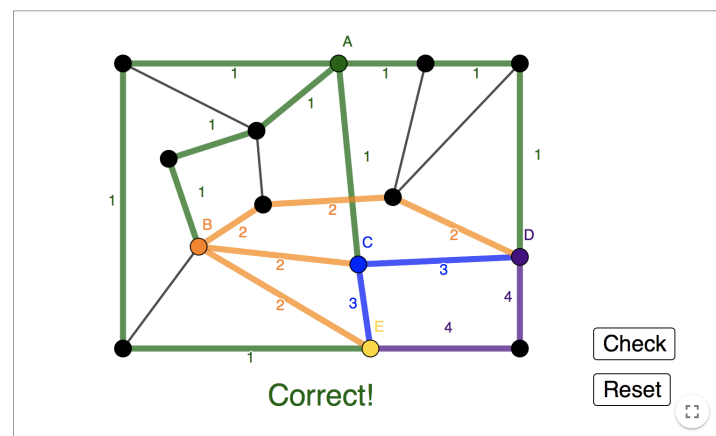
In other words, we can show that a graph is non-planar by finding either a $K_{3,3}$ or K_5 edge subdivision within the graph.

Watch this video: <https://youtu.be/OCegRy-xhjQ> for examples of using Kuratowski's Theorem to prove a graph is non-planar. Then, use the following online activities to practice finding a $K_{3,3}$: <https://www.geogebra.org/m/khnvzzmn> and a K_5 : <https://www.geogebra.org/m/ufzpbfvy> edge subdivision, thus proving by Kuratowski's Theorem that the graphs are non-planar.

$K_{3,3}$ Solution:



K_5 Solution:

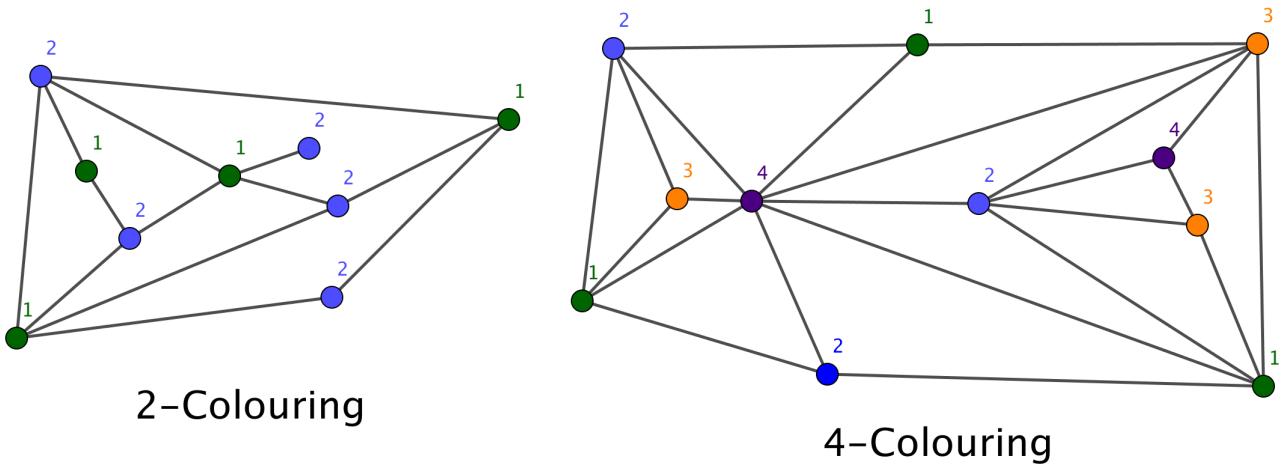


Colouring

Definition 7. k -Colouring

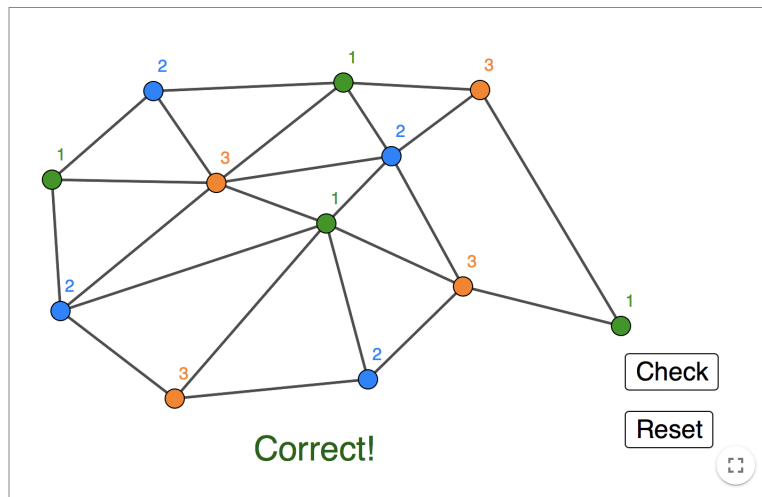
A k -colouring of a graph is an assignment of at most k different colours so that each vertex is assigned a colour. Each vertex must have a different colour than any of its neighbours.

Examples of colourings:



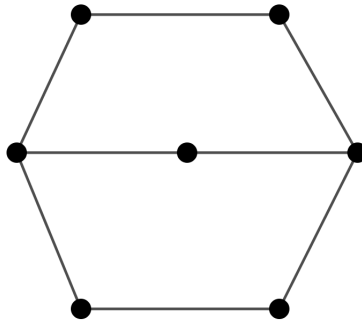
Exercise 3. Find a 3-colouring: <https://www.geogebra.org/m/eddgdxh5>

Solution:



An interesting question is the least number of colours that we can colour a graph with. This is called a graph's **chromatic number**. We can prove that something is k -colourable by showing an example of a colouring. But how do we show that something is not k -colourable?

Consider the following graph:



It is not 2-colourable.

Use this online activity: <https://www.geogebra.org/m/vu3rgygg> to convince yourself of this.

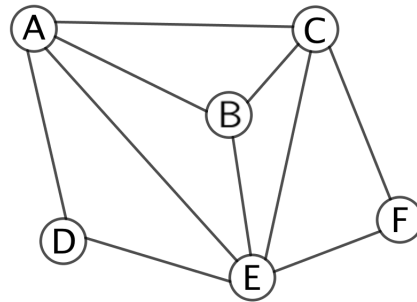
Think about what makes this graph not 2-colourable.

Watch the video: <https://youtu.be/r6VhIWYFVyw> for a full explanation.

Problem Set

1. For the graph to the right:

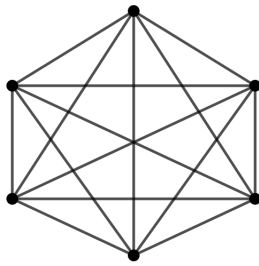
- (a) Find a path from vertex D to vertex C.
- (b) List the neighbours of vertex B.
- (c) Find a 4-colouring.
- (d) Explain why a 3-colouring can't be found.



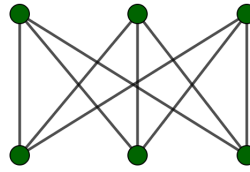
Solution:

- (a) One possibility: $D, \{D, A\}, A, \{A, C\}, C$
 - (b) A, C, E
 - (c) Colour A and F the first colour and D and C the second. Colour B the third and E the fourth.
 - (d) A 3-colouring cannot be found as there are 4 vertices in the graph that are all connected to one another (A, B, C and E). Since they are all connected, we have found a K_4 subgraph and any colour used by A could not be used by B, C or E . It is similar for the other vertices. Thus, the 4 vertices all must have unique colours. So, we need to have at least 4 colours to colour that portion of the graph. Thus, a 3-colouring can't be found for the graph.
2. For each part listed below, draw a possible graph with the given properties:
- (a) Exactly 6 vertices and 15 edges.
 - (b) A 2-colouring, exactly 6 vertices and 9 edges.
 - (c) Non-planar.
 - (d) No 2-colouring.
 - (e) No 6-colouring.

Solution:



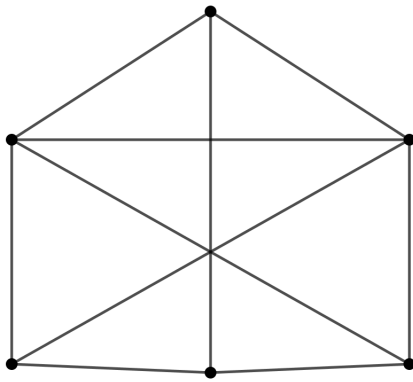
a)



$K_{3,3}$

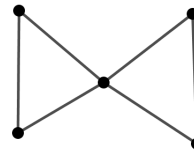
b)

A 2-colouring of this graph could be shown by colouring the top three vertices one colour and the bottom three another.



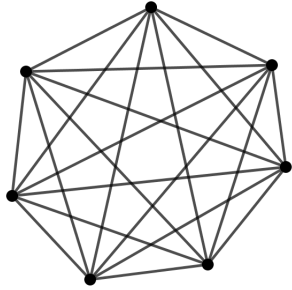
c)

This graph is non-planar as it has a $K_{3,3}$ edge subdivision. Notice how the top, the bottom left and the bottom right vertices all have edges to the other three vertices. This is the edge subdivision. So, by Kuratowski's Theorem, this graph is non-planar.



d)

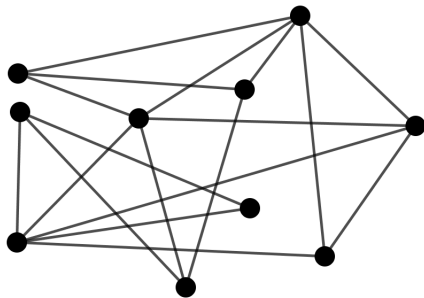
This graph doesn't have a 2-colouring as it has an odd cycle (both the left and right sides show cycles of length 3). As we discussed earlier, this means that the graph does not have a 2-colouring.



e)

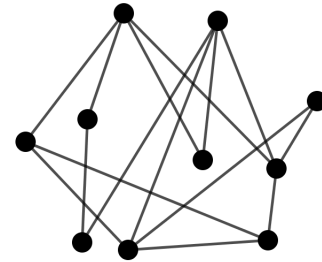
Notice how each of the 7 vertices connects to each other vertex. So, this graph is the K_7 and if I choose any vertex, it cannot be the same colour as any of the others. Since this is true for all the vertices, we need to have each vertex be a different colour. Since there are 7 vertices, the minimum colouring we could have is a 7-colouring. So, there is no 6-colouring.

3. Prove that the following graphs are planar. *Hint: Click on the graph names to access a movable graph.*



Graph 1:

<https://www.geogebra.org/m/es5vpja2>



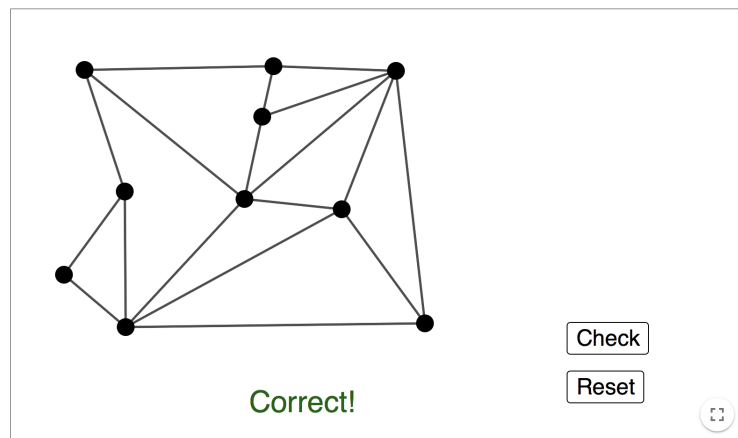
Graph 2:

<https://www.geogebra.org/m/qkjuvfkq>

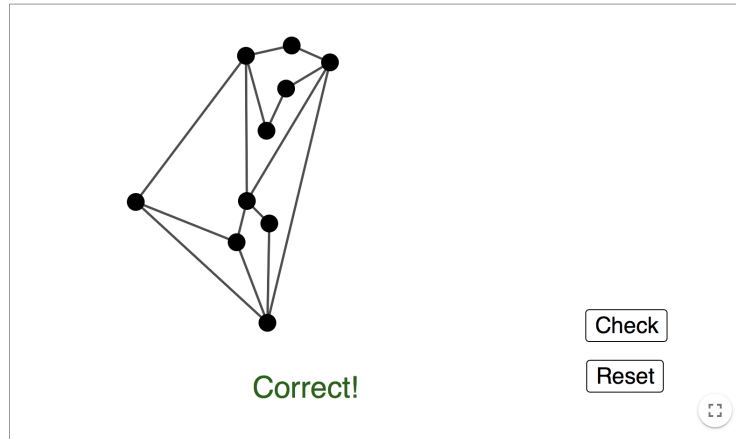
Solution:

To prove that the graphs are planar, we give planar embeddings.

For Graph 1:

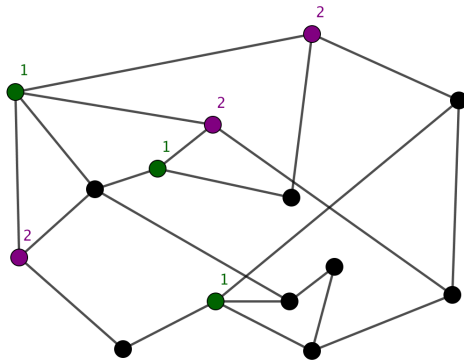


For Graph 2:

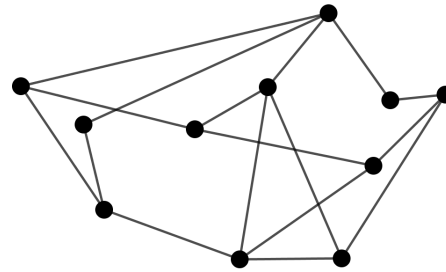


4. Find an edge subdivision of a $K_{3,3}$ in the following graphs to prove that they are non-planar. For the first graph, a recommended 6 vertices, split into 2 groups, has been given.

Graph 1

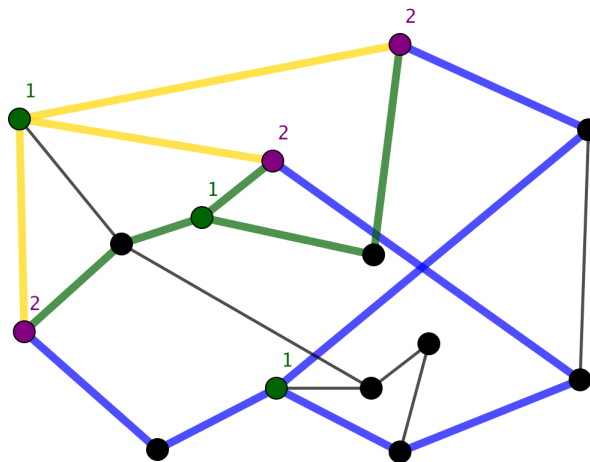


Graph 2

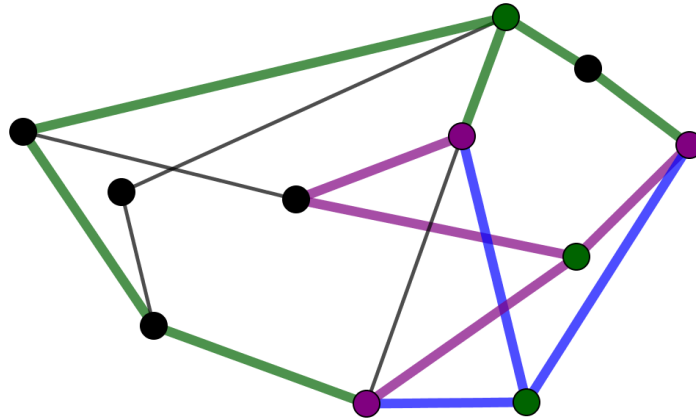


Solution:

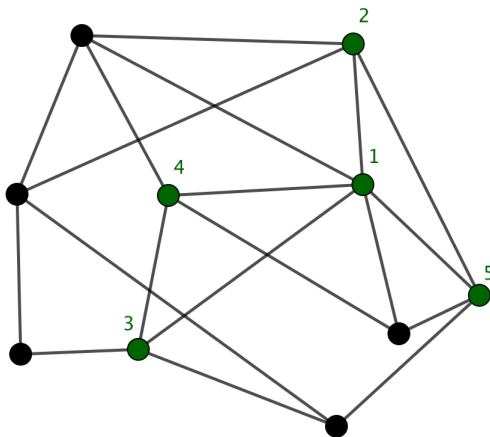
For Graph 1:



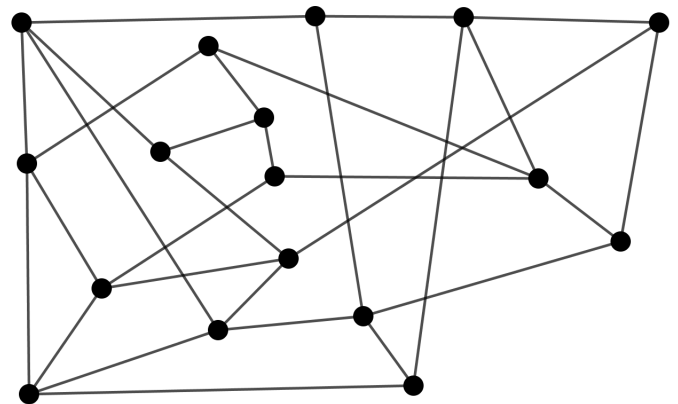
For Graph 2:



5. Find an edge subdivision of a K_5 in the following graphs to prove that they are non-planar. For the first graph, a recommended 5 vertices has been given.



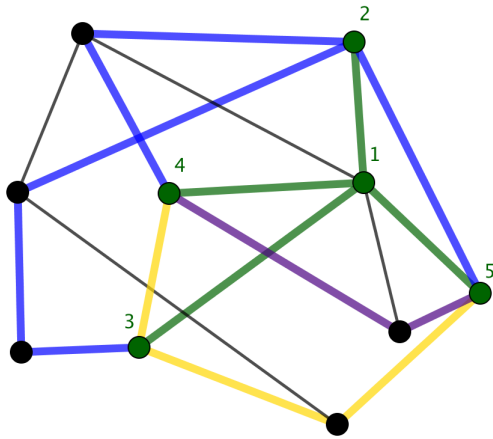
Graph 1



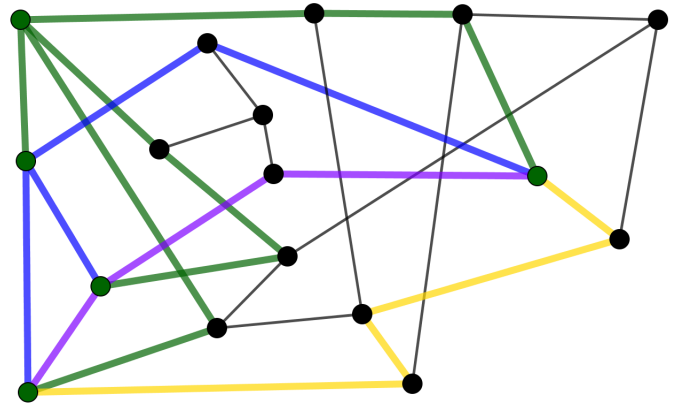
Graph 2

Solution:

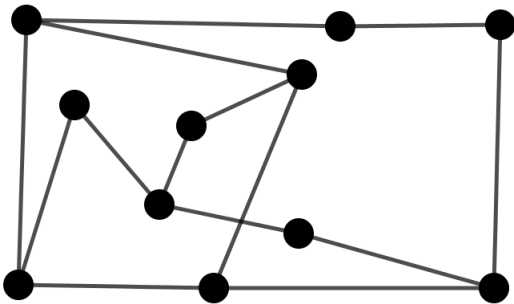
For Graph 1:



For Graph 2:

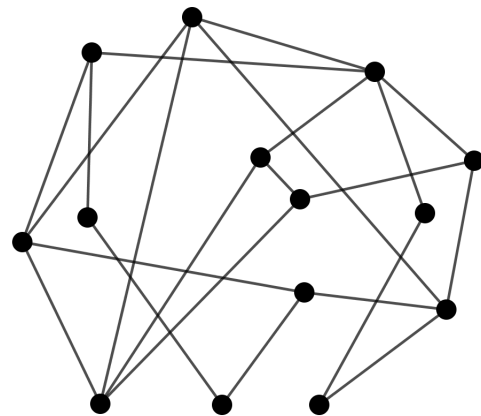


6. Prove that the following graphs are non-planar.



Graph 1

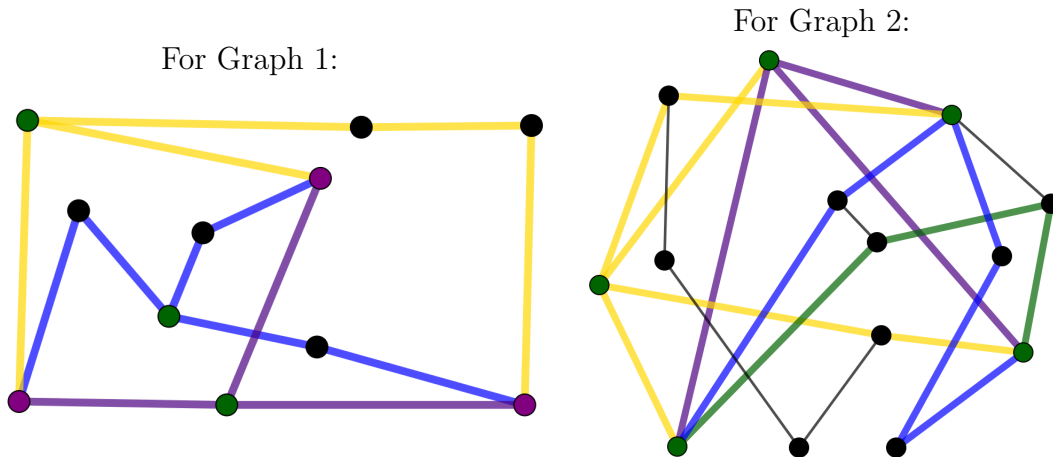
Our example graph from earlier



Graph 2

Solution:

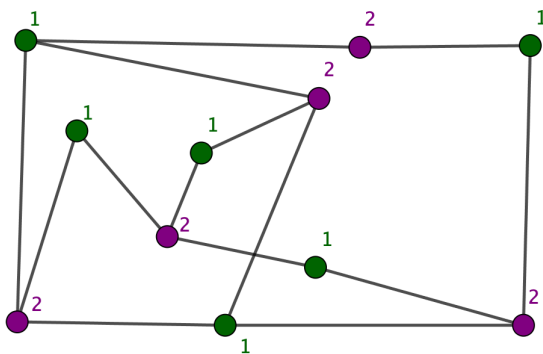
To do this, we can use Kuratowski's Theorem and find either a $K_{3,3}$ or K_5 edge subdivision of the graphs. For the first graph, we find a $K_{3,3}$ and for the second a K_5 .



7. Find the chromatic number for the graphs in problem 6. Prove that the number you find is indeed the chromatic number. *Hint: To prove that a chromatic number, k , is correct, we need to show that that the graph is k -colourable and that it is not $(k - 1)$ -colourable.*

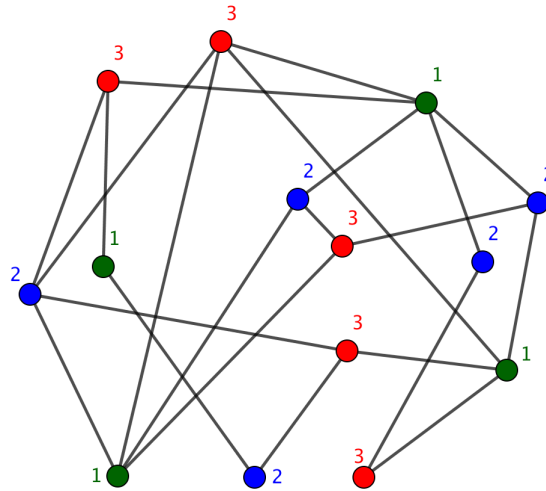
Solution: To prove that a number, n , is the chromatic number for a given graph, we need to show that an n -colouring exists for the graph and that an $(n - 1)$ -colouring doesn't.

For the first graph, we can create a 2-colouring as seen below. We cannot have a 1-colouring as there is an edge in the graph and any two vertices connected by an edge must have different colours. Thus, the chromatic number is 2.



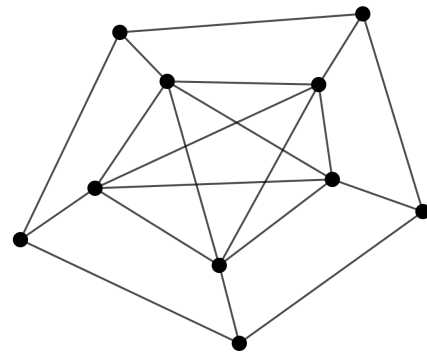
For the second graph, we can create a 3-colouring as seen below. We cannot have a 2-colouring as we have an odd cycle in the graph (as an example, we can see a cycle of length three near the middle of the graph). When there is an odd cycle, we can, without loss of generality, choose a vertex to start at. The next vertex in the cycle must have a different colour. The next from that may have the initial colour, and so

on. However, since the number of vertices in the cycle is odd, the first vertex and the last vertex prior to reaching the start again will need to have the same colour, which cannot be in a valid colouring. Thus, this graph cannot have a 2-colouring, as odd cycles cannot be coloured by only two colours, and has a chromatic number of 3.



8. Prove that the graph to the right is not 4-colourable.

Hint: Find a K_5 graph as a part of the graph (Not an edge subdivision). Explain why this means it is not 4-colourable.



Solution:

Notice that the middle five vertices form a K_5 graph. As a result, they each connect to all others in the K_5 . When choosing an arbitrary vertex from those five, it will need to have a different colour than any of the other four. Since this applies to each of the five vertices, we need five colours to colour those five vertices. Thus, we need at least five colours to colour the graph and hence the graph is not 4-colourable.

9. *Timetabling (BCC 2018)*

This is one of the BCC problems that we tried last week. This week, let's try to use what we have learned about colouring to answer the problem.

Bebras Tech offers the following evening classes: Computing (**C**), Geography (**G**), Language (**L**), Math (**M**), and Science (**S**).

Three beavers would like to sign up for these courses: Xavier wants to take **C**, **L** and **M**; Yvette wants to take **C**, **G**, and **S**; Zoey wants to take **L**, **M** and **S**.

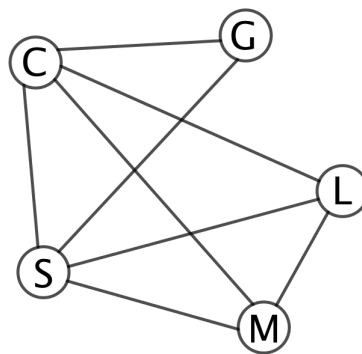
Bebras Tech wants to squeeze these courses into as few evenings as possible such that each course is offered on exactly one evening, and beavers can take at most one course per evening.

What is the least number of evenings needed for Bebras Tech to schedule these courses?

Hint: Let each course be represented by a vertex. Represent each possible conflict as an edge. Assigning a vertex a colour is then the same as assigning a course a night.

Solution:

Below, we can see the resulting graph from assigning each vertex to a course and each possible conflict to an edge. Notice that vertices C , L , M and S are all connected with each other, forming a K_4 . Thus, any colour that we choose for one of those vertices cannot be the same as that of the others. Since this holds for all four of the vertices, we will need at least four colours to colour the graph. If we colour each a different colour, then L has a colour different from both C and S , so G can be coloured the same as L . Thus the graphs chromatic number is 4, which means that the minimum number of nights is 4, same as the result we found last week.



10. Word of your skills in timetabling has gotten around and the university has asked for your help in scheduling the following courses, with conflicts represented as X's in the below table:

Each course is to be assigned one time slot. Represent this situation with a graph and use your knowledge of colouring to find and prove what the minimum number of timeslots would be.

Table 1: Course Conflicts

	Algebra	Computing	Calculus	English	French	Physics
Algebra		X	X	X		X
Computing	X				X	X
Calculus	X			X		X
English	X		X		X	
French		X		X		
Physics	X	X	X			

Solution:

We will solve this problem similar to problem 9. First, we create a graph that represents the problem, with the courses as vertices and the possible conflicts as edges. Then, we will determine the chromatic number of the graph. This in turn will represent the minimum number of timeslots required to schedule the courses. Below, we have the resulting graph from the translation of this problem. One way to colour the below graph is to have Computing and Calculus be one colour, Physics and English another and Algebra and French be a third. So, we know that the graph is 3-colourable. To prove that 3 is its chromatic number, we also need to show that it is not 2-colourable. Notice that Algebra, English and Calculus are all connected to each other, forming a K_3 . Thus, if we colour one, the other two cannot be that colour. Since this is true for all three vertices, they all must have different colours. Thus, the graph is not 2-colourable and therefore 3 is the chromatic number. Thus, the minimum number of timeslots required is 3.

