Intermediate Math Circles Fall 2020

Counting

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Counting Integers

We are going to look at four problems and their solutions to introduce some ways of counting of integers.

Problem 1

How many two-digit integers can you make using only the digits 6, 7, or 8? (Note: Two examples are 67 or 88)

Take a few minutes to try this question before looking at the solution on the next few pages.

Problem 1 Solution

There are two possible solutions.

Solution 1:

We can list all the possibilities. They are :

66, 67, 68, 76, 77, 78, 86, 87, 88.

Therefore, there are nine two-digit integers can you make using the digits 6. 7, or 8.

Problem 1 Solution

Solution 2:

We will look at the digits of the possible integers.

There are 3 choices for the first digit and 3 choices for the second digit.
This means there are $3 \times 3 = 9$ two-digit integers you can make using the digits 6. 7, or 8.

(This second solution will work best when the answer is large.)

Counting Problem 2

If you wrote down all of the integers from 300 to 599, inclusive, how many times would you write the digit 4?
Inclusive means that both 300 and 599 are included in the list.

Take a few minutes to try this question before looking at the solution on the next few pages.

Counting Problem 2 Solution

To answer, you need to count how many times you would write the digit 4 when writing the integers from 300 to 599, inclusive. Since there would be 300 integers in this list, (can you show that there are 300 integers from 300-599 inclusive?) actually writing them all down is not the best approach. There are a number of ways to approach this question, but we will use an organized count.

Counting Problem 2 Solution

First consider the units digit.
For every cycle of the 10 digits from 0-9, the digit 4 occurs once.
To find the number of cycles of these 10 digits, we will take the total number of integers and divide by the length of the cycle.
There are $300 \div 10 = 30$ cycles of the digits 0-9. Therefore there are 30 integers with a 4 in the units digit when writing the integers from 300 to 599.

Counting Problem 2 Solution

Next, consider the tens position.
For every cycle of the 100 digits 00-99, the digit 4 will occur 10 times in the tens position. (These are in the integers 40 - 49.)
There are $300 \div 100 = 3$ cycles of 00-99. This means there are $10 \times 3 = 30$ integers that have the digit 4 in the tens position when writing the integers from 300 to 599.

Counting Problem 2 Solution

Finally, consider the hundreds positions.
There are 100 integers that have a 4 in the hundreds position when writing the integers from 300 to 599. (These are 400-499).

Counting Problem 2 Solution

Combining the counts for the three different positions, when writing the integers from 300 to 599 the digit 4 will be written a total of $30 + 30 + 100 = 160$ times.

Counting Problem 3

How many of the integers from 300 to 599, inclusive, contain the digit 4 at least once?

Take a few minutes to try this question before looking at the solution on the next few pages.

Counting Problem 3 Solution 1

In our solution to Problem 2, we determined that there are 100 integers from 300 to 599 with a 4 in the hundreds position, 30 integers with a 4 in the tens position, and 30 integers with a 4 in the units position, but we need to note that $100 + 30 + 30 = 160$ is not a correct count of the number of integers that contain at least one 4.

Some of these integers have more than one digit equal to 4 and so were counted more than once in the total of 160.

Counting Problem 3 Solution 1

We now need to determine how many integers were counted more than once, and exactly how many times they were counted. Any integer with more than one 4 will be counted more than once.

How many of the integers have a 4 in both the hundreds and the tens positions? These integers will begin with 44, and thus are from 440 to 449, and there are 10 of them. These 10 integers have been counted at least twice.

How many of the integers have a 4 in both the hundreds and the units positions? These integers will begin with a 4 and end with a 4. These integers have the form 4X4, where the X can be any of the ten digits, and there are 10 of them. These 10 integers have been counted at least twice.

Counting Problem 3 Solution 1

How many of the integers have a 4 in both the tens and the units positions? These integers will end with 44. They are 344, 444 and 544.

To count the number of integers that contain at least one digit 4 we do the following calculation. Start with the total of 160 and subtract the number of integers that were counted at least twice (due to having a digit 4 in at least two positions). This results in the following:

$$160 - 10 - 10 - 3 = 137$$

But there is one last thing to consider before we obtain the final answer.

Counting Problem 3 Solution 1

The integer 444 was counted (or included) three times and then removed (or excluded) three times in our calculation above. (Do you see why?) This means it is not included in the count of 137. Adding the integer 444 back into our count we get a final answer of

$$160 - (10 + 10 + 3) + 1 = 138$$

Therefore, there are 138 integers between 300 and 600 that contain the digit 4 at least once.
Note: In this solution, the method of *inclusion-exclusion* has been demonstrated.

Counting Problem 3 Solution 2

Here is a second way to solve this problem. There are two cases we need to look at:

- **Case 1:** The integer contains the digit 4 at least once.
- **Case 2:** The integer doe not contain the digit 4.

We will find the number of the integers from 300 to 599, inclusive, that contain the digit 4 in a different way from solution 1.

We will count the number of integers that do not contain the digit 4 and subtract this number from the total number of integers from 300 to 599.

Counting Problem 3 Solution 2

Now, we will count the number of integers that do not contain the digit 4.

There are two choices for the first digit. They are 3 or 5.
There are nine choices for each of the following two digits. They are 0, 1, 2, 3, 5, 6, 7, 8, or 9.

This means the number of integers from 300 to 599, inclusive, in which the digit 4 does not occur is $2 \times 9 \times 9 = 162$.

Since the total number of integers is 300, the total number of integers from 300 to 599, inclusive, that contain the digit 4 at least once is :

$$300\text{-}162 = 138$$

Counting Problem 4

How many of the integers from 300 to 599, inclusive, have the digit 4 occur exactly once?

Take a few minutes to try this question before looking at the solution on the next few pages.

Counting Problem 4 Solution

There are three cases we need to look at.

- **Case 1:** The first digit is 4 and the other two are not 4.
- **Case 2:** The second digit is 4 and the other two are not 4.
- **Case 3:** The third digit is 4 and the other two are not 4.

Counting Problem 4 Solution

Case 1: The first digit is 4 and the other two are not 4.

If the first digit is 4, there are nine choices for the other each other digit. These digits are 0, 1, 2, 3, 5, 6, 7, 8, or 9.
This means that there are $1 \times 9 \times 9 = 81$ integers where the first digit is 4 and the other two are not 4.

Counting Problem 4 Solution

Case 2: The second digit is 4 and the other two are not 4.

If the second digit is 4, there are two choices for the first digit. It can be 3 or 5. There are nine choices for the third digit.
This means that there are $2 \times 1 \times 9 = 18$ integers where the second digit is 4 and the other two are not 4.

Counting Problem 4 Solution

Case 3: The third digit is 4 and the other two are not 4.

If the third digit is 4, there are two choices for the first digit.
There are nine choices for the second digit.
This means that there are $2 \times 9 \times 1 = 18$ integers where the third digit is 4 and the other two are not 4.

Counting Problem 4 Solution

We now need to add up all the cases:

$$81 + 18 + 18 = 117$$

Therefore, there are 117 integers from 300 to 599, inclusive, that have the digit 4 occur exactly once.

Your Turn

Now try the exercises given in the problem set.