Intermediate Math Circles Fall 2020

Triangles, Circles and Area

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In this session, we are going to be looking at triangles, circles and the area of certain parts of these circles. We will do this in the following sections:

1. Isosceles Triangle Properties
2. Area of an Equilateral Triangle
3. Problem 1
4. Problem 2

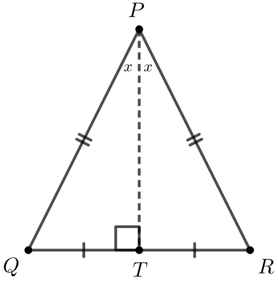
Isosceles Triangle Properties

We will use the following fact without proving it.

When a perpendicular is dropped from a vertex to the unequal side of an isosceles triangle, it bisects the unequal side and it bisects the unequal angle.

In the diagram to the right $\triangle PQR$ is an isosceles triangle with $\angle PQR = \angle PRQ$ and $PQ = PR$.

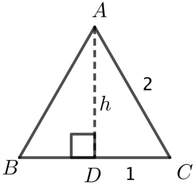
Construct T on QR such that PT is perpendicular to QR. We now have PT bisecting QR. This means $QT = RT$. We also have PT bisecting $\angle QPR$. This means $\angle QPT = \angle RPT$.



Note that an equilateral triangle is a special isosceles triangle. This means this fact is true when we drop a perpendicular from any vertex of the equilateral triangle to *its opposite side.*

Area of an Equilateral Triangle

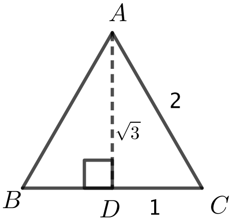
Let's look at equilateral $\triangle ABC$ with a side length of 2.
Drop a perpendicular from A to BC meeting BC at D. This is the height, h.
Since $\triangle ABC$ is equilateral, D bisects side BC and $DC = 1$.



Area of an Equilateral Triangle

Now we will use $\triangle ADC$ and the Pythagorean Theorem.

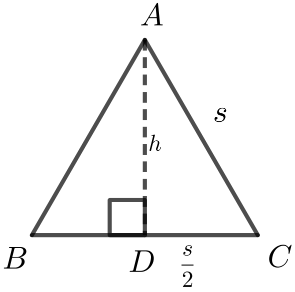
$$\begin{aligned} AD^2 &= AC^2 - DC^2 \\ h^2 &= 2^2 - 1^2 \\ &= 4 - 1 \\ &= 3 \\ h &= \sqrt{3}, \text{ since } h > 0 \end{aligned}$$



The area of $\triangle ABC = \frac{bh}{2} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

Area of an Equilateral Triangle

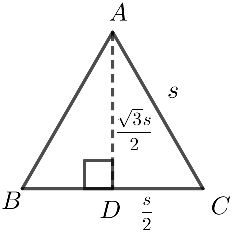
In general, let's look at equilateral $\triangle ABC$ with a side length of s. We will drop a perpendicular from A to BC meeting BC at D. Therefore, $DC = \frac{s}{2}$.



Area of an Equilateral Triangle

Now we will use $\triangle ADC$ and the Pythagorean Theorem.

$$\begin{aligned} AD^2 &= AC^2 - DC^2 \\ h^2 &= s^2 - \left(\frac{s}{2}\right)^2 \\ &= s^2 - \frac{s^2}{4} \\ &= \frac{3s^2}{4} \\ h &= \frac{\sqrt{3}s}{2}, \text{ since } s > 0 \text{ and } h > 0 \end{aligned}$$



The area of equilateral $\triangle ABC = \frac{bh}{2} = \frac{1}{2}(s\frac{\sqrt{3}s}{2}) = \frac{\sqrt{3}s^2}{4}$.

Triangles, Circles and Areas Problems

To summarize:

Given equilateral $\triangle ABC$ with side length, s:

the area of $\triangle ABC = \frac{\sqrt{3}s^2}{4}$.

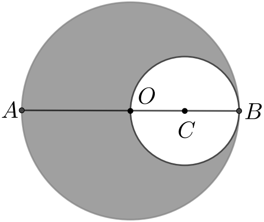
(This is a good formula to have in your math toolbox.)

Triangles, Circles and Areas Problems

For each of the following problems, use the information and diagram given to **find the area of the shaded region**. Express your answers as simplified exact numbers. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Problem 1

Two circles are centred at C and O as shown. AB is a diameter of the larger circle. OB is a diameter of the smaller circle. The larger circle has a diameter of 20. Find the area of the shaded region.



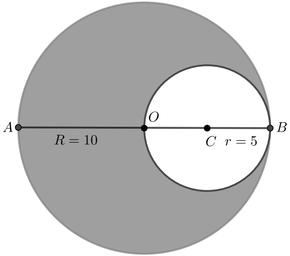
Take a few minutes to try this question before looking at the solution on the next page.

Problem 1 Solution

We will find the area of the shaded region by subtracting the area of the smaller circle from the area of the larger circle.

Let R be the radius of the larger circle and r be the radius of the smaller circle.

Since the diameter of the smaller circle is the radius of the larger circle, we have $R = \frac{20}{2} = 10$ and $r = \frac{10}{2} = 5$.



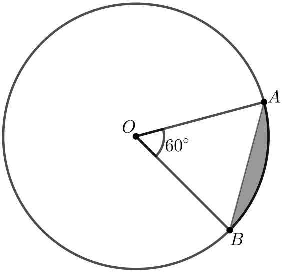
$$A_{\text{larger}} = \pi R^2 = \pi(10)^2 = 100\pi$$
$$A_{\text{smaller}} = \pi r^2 = \pi(5)^2 = 25\pi$$

Therefore, the area of the shaded region is $100\pi - 25\pi = 75\pi$.

Problem 2

The circle with centre O has a radius of 20. Points A and B are on the circle and $\angle AOB = 60^\circ$ as shown. Find the area of the shaded region.

Hint: Show that the triangle is an equilateral triangle.



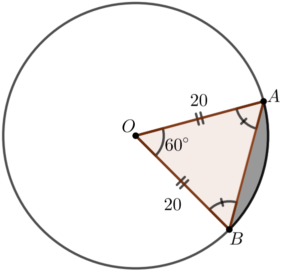
Take a few minutes to try this question before looking at the solution on the next few pages.

Problem 2 Solution

To find the area of the shaded region, we will find the area of the sector of the circle with arc AB and subtract the area of $\triangle AOB$.

We will first show that $\triangle AOB$ is equilateral. Note that the $\triangle AOB$ is an isosceles triangle with sides OA and OB equal to the radius of 20. Therefore, $\angle OAB = \angle OBA$. We also know that the sum of the angles in a triangle is 180°. Therefore:

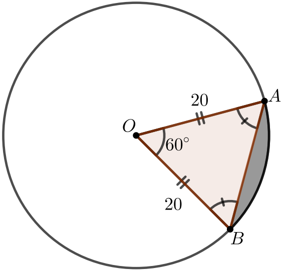
$$\begin{aligned} \angle OAB + \angle OBA + \angle AOB &= 180 \\ \angle OAB + \angle OAB + 60 &= 180 \\ 2\angle OAB &= 120 \\ \angle OAB &= 60 \end{aligned}$$



Therefore, $\angle OAB = \angle OBA = \angle AOB = 60^\circ$ and $\triangle AOB$ is an equilateral triangle with a side length of 20.

Problem 2 Solution

We will now find the two areas and then subtract the area of triangle from the area of the sector. To find the area of the sector we will find the fraction of the circle (remember there are 360° in a circle). To find the area of the triangle we will use the formula for an equilateral triangle found earlier.



$A_{wholeCircle} = \pi r^2 = \pi(20)^2 = 400\pi$

$A_{sector} = \left(\frac{60}{360}\right) 400\pi = \frac{200\pi}{3}$

$A_{equilateraltriangle}$

$= \frac{\sqrt{3}s^2}{4} = \frac{\sqrt{3}(20)^2}{4} = 100\sqrt{3}$

Therefore, the area of the shaded region is $\dfrac{200\pi}{3} - 100\sqrt{3}$.

You can now try the problem set given.