UNIVERSITY OF WATERLOO

Faculty of Mathematics
Waterloo, Ontario N2L 3G1

Centre for Education in
Mathematics and Computing

Grade 6 Math Circles

October 28, 2020

Counting Part II - Solutions

By now you should be a pro at using the Fundamental Counting Principle and solving basic problems using the Permutation formula. In this lesson, we will build on our knowledge of Permutations and introduce another method of counting called **Combinations!**

Permutations with Repeats

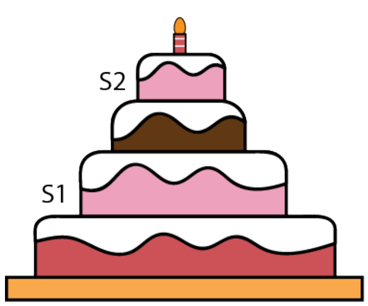
Example: Bake a Cake continued

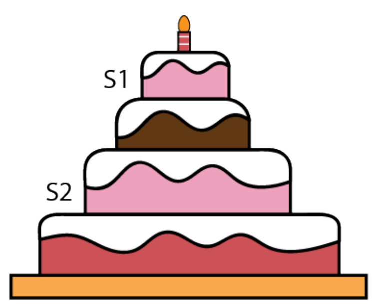
Suppose you are making a four-layer cake and have four flavours of cake to choose from: **strawberry, chocolate, strawberry, and red velvet**. Each layer of this cake must be a different flavour. How many different four-layer cakes can you make? Previously we saw:

$$\text{total number of different cakes} = 4! = 4 \times 3 \times 2 \times 1 = 24$$

Is this correct? You may have noticed that we list the flavour strawberry twice. Does this change how we previously solved this problem?

Yes! - Let's take a further look. In this example, each four-layer cake will have two layers that are strawberry. To help us understand this example let's call one of the strawberry layers **S1** and the second strawberry layer **S2**. Consider the following cakes:





If you look at the two cakes, there are two layers of strawberry in each. Both starwberry layers are identical: we can put S1 as the second layer in our cake, and S2 as the fourth layer, or we can put S2 as the second layer in our cake, and S1 as the fourth layer. It **does not matter** which order we put these strawberry layers in since they give us the exact same cake.

This means that for every four-layer cake we make, we can simply swap the two flavours (S1 and S2) and get the exact same cake again. Because of this, we must account for the fact that we are counting both cakes, when we should just be counting one of them since they are identical!

Permutations with Repeats:
When we have n items and want to know how many ways we can order all n items given that there are r repeats of an item or several items, then:

$$\text{One Repeat: } \frac{n!}{r!} \text{ or Multiple Repeats: } \frac{n!}{r_1!r_2! \ldots r_n!}$$

Here r_1 represents the fact that item 1 is repeated r_1 times, r_2 represents the fact that item 2 is repeated r_2 times, and so on. Let's apply this formula to the above example.

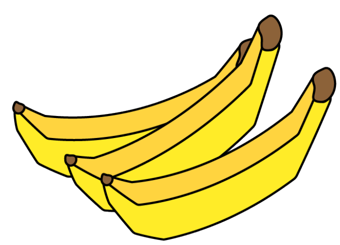
Solution: Here the numerator (4!) is simply from the numerator of our previous answer where there were no repeats. But since we have 2 repeats of the strawberry flavour:

$$\text{total number of different cakes} = \frac{4!}{2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1} = 12$$

By dividing by two, we essentially "got rid" of the extra identical cakes because we want the total number of **different** cakes.

Example: Using the letters in the word: **BANANA**

1. How many different 6-letter "words" can be formed?



Note: These do not have to be real words.

BANANA Example Video Solution:
https://youtu.be/aZli01zeyts

First let's take a look at how many letters we have in total. How many letters are repeated?

- We have 6 letters in total in the word BANANA.
- We have 3 repeats of the letter 'A'.
- We have 2 repeats of the letter 'N'.

Since we have 6 letters in total and we want to arrange all 6 letters, with 2 repeats of the letter 'N', and 3 repeats of the letter 'A', we know this is a Permutation with Repeats problem! Here $n = 6$, $r_1 = 2$, and $r_2 = 3$. Thus using the Permutations with Repeats formula:

$$\begin{aligned}\text{total number of different 6-letter words} &= \frac{6!}{(2!)(3!)(6-6)!} \\ &= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{(2 \times 1)(3 \times 2 \times 1)0!} \\ &= \frac{6 \times 5 \times 4}{2} \\ &= 60\end{aligned}$$

Therefore there are 60 different 6-letter words using the letters of BANANA.

How Many Orders?

In problems about counting we often want to ask ourselves whether or not "order matters" in our choices. What is order in this case? And what does it mean? Let's take a look at two examples that can help us understand what "order" is.

Example 1: Ryan, Tim, Vince and Luc are in the finals for a race at the UW track meet. How many different ways can we award a first, second, and third place prize for the race? List all of the possible ways that the top 3 prizes can be awarded.

Solution: Let's write out all possible ways to award the top 3 prizes as "First Second Third" where R represents Ryan, T - Tim, V - Vince, L - Luc:

$R\ L\ V$	$R\ V\ L$	$R\ L\ T$	$R\ T\ L$	$R\ T\ V$	$R\ V\ T$
$L\ R\ V$	$L\ V\ R$	$L\ V\ T$	$L\ T\ V$	$L\ T\ R$	$L\ R\ T$
$V\ R\ T$	$V\ T\ R$	$V\ L\ T$	$V\ T\ L$	$V\ L\ R$	$V\ R\ L$
$T\ R\ V$	$T\ V\ R$	$T\ L\ R$	$T\ R\ L$	$T\ L\ V$	$T\ V\ L$

Thus, there 24 different ways to award a first, second, and third place prize.

Example 2: A national track team wants to make a team from Ryan, Tim, Vince and Luc at UW. How many different three-man teams can be made?

Solution: Let's write out all possible teams that can be made using the four athletes:

$R\ L\ V$	$R\ L\ T$	$R\ V\ T$	$V\ T\ L$

Thus, there are 4 different teams that could be made of only three athletes.

In both questions we looked at taking three students from four available students. But while they seem to follow the same idea, we ended up with different answers!

Let's take a closer look at the two examples and see how the order of their items affected the answer:

- **Example 1:**

  In the race there is a first, second, and third place prize. If the final outcome was R-T-V, this means that Ryan came first, Tim came second, and Vince came third. R-V-T would NOT be the same as R-T-V because now Vince came second, and Tim came in third. We can say that **ORDER MATTERS**.

- **Example 2:**

  In constructing a three-man team, if one of the athletes made the team, then they're on the team. There is no placing or hierarchy on the team. This means that the team R-V-T is the same as R-T-V, T-V-R, T-R-V, V-R-T and V-T-R. We can say that **ORDER DOESN'T MATTER**.

This is actually the difference between **Permutations** and **Combinations**!

In Permutations **order matters**

In Combinations **order doesn't matter**

Knowing when order does and doesn't matter is an important part of working with Permutations and Combinations so let's get some practice before we introduce Combinations.

Example: For each of the following scenarios, state whether or not order matters:

1. The number of ways three distinct plants can be arranged on a window sill.

   ORDER MATTERS

2. Mr. Elgoog is asked to draw three cards from a deck of cards. In how many ways can he select three cards?

   ORDER DOESN'T MATTER

3. A math student is given a list of 8 problems and is asked to solve any 5 of the problems. How many different selections can the student make?

   ORDER DOESN'T MATTER

4. Selecting a 4-digit PIN code for a credit card.

   ORDER MATTERS

5. The National Quidditch team is holding tryouts for one Keeper, one Seeker, and one Chaser. Seven students try out. How many ways can the three positions be filled?

   ORDER MATTERS

Orders Example Video Solution: *https://youtu.be/NLYMW1E5l-g*

Note: In all of the Permutation problems we solved last week, order mattered!

Combinations

What if we had n objects in total and needed to choose k with **no repetition** (just like in a permutation) but now **order does not matter**?

> **Combinations** are a way of counting in this type of situation when there is no repetition and **order doesn't matter**.
>
> $$ {}_nC_k = \frac{n!}{k!(n-k)!} $$
>
> This is read as "n choose k" and counts how many ways we can choose k objects from a total of n objects.

We divide $n!$ by $k!$ because there are $k!$ ways to order the chosen k objects when order doesn't matter. If we do not divide by $k!$, we are counting each combination $k!$ times and we will not have the correct solution. We also divide $n!$ by $(n-k)!$ because again, we do not want to count the remaining $(n-k)$ objects that were not selected (we saw this in the Permutation formula as well).

Note: We can also write ${}_nC_k = \binom{n}{k}$

Example: Chocolate Frog Cards

Harry loses a bet to Ron and has to give Ron **4** of his chocolate frog **cards**. If Harry has **49 different cards** in his collection, how many ways can Ron pick which cards he'll take?

1. How many cards in total do we have to choose from and how many do we need to choose?

2. Does order matter when we choose our cards? Why?

3. Is repetition allowed in our choices? Why?

4. Use the combination formula to find how many ways Ron can pick the cards.

Solution:

1. There are 49 cards in total to choose from and we need to choose 4.
2. No, order does not matter because Ron ends up with the same 4 cards no matter which he picks first, second, third, and fourth.
3. No, there is no repetition allowed because once a card is picked, it can't be picked a second time.
4. Since **order doesn't matter** this is a Combinations problem. There are 49 total cards to choose from so $n = 49$. Since Ron needs to choose 4 cards, $k = 4$. Using the Combinations formula we get:

$$_{49}C_4 = \frac{49!}{4!(49-4)!} = \frac{49!}{4!45!} = \frac{49 \times 48 \times 47 \times 46}{4!} = 211,876$$

   Thus there are 211, 876 ways Ron can pick the cards.

Example: Student Committee

There are 16 students on your school's student committee and there are 3 available executive positions. How many different ways can 3 students be elected from the student committee?

Solution:

We are electing 3 students from the student committee of 16 students thus $n = 16$ and $k = 3$. The order in which these students are elected does not matter so we use the Combinations formula:

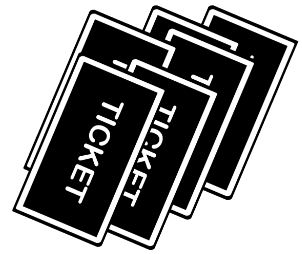
$$_{16}C_3 = \frac{16!}{3!(16-3)!} = \frac{16!}{3!13!} = \frac{16 \times 15 \times 14}{3!} = 560$$

There are **560** different ways to elect 3 students from the student committee.

Example: Movie Night

Suppose you just won 7 free movie tickets. You want to bring along 6 friends, but unfortunately, you have 10 friends who want to come along!

1. Does order matter?
2. Is repetition allowed?
3. How many different groups of friends could you take with you?



4. To make it a little fair, you decide you want to bring along exactly 3 girls and 3 boys. From the group of 10 friends who want to come along, there are 4 boys and 6 girls. How many ways can you pick 3 boys AND 3 girls?

Movie Night Example Video Solution: *https://youtu.be/jOQpA-zAx_U*

1. Consider the choice written as (Person 1, Person 2, Person 3, Person 4, Person 5, Person 6). Is (Alice, Emily, Jack, John, Allison, Patty) different from (John, Patty, Jack, Alice, Allison, Emily)? No! - Since we get the same group of friends no matter who we pick first, second, third, fourth, fifith, and sixith. So order does not matter!
2. No! - Repetition is not allowed. Once a friend is picked, they can't be picked again.
3. Since **order doesn't matter** and there is no repetition this is a Combinations problem. Recall: ${}_nC_k = \dfrac{n!}{k!(n-k)!}$. There are 10 friends to choose from, then $n = 10$. We need to choose 6 friends to take to the movies, then $k = 6$. Using the Combinations formula we get:

$$\begin{aligned} {}_{10}C_6 &= \frac{10!}{6!(10-6)!} \\ &= \frac{10!}{6!4!} \\ &= \frac{10 \times 9 \times 8 \times 7}{4!} \\ &= 210 \end{aligned}$$

   Thus there are 210 different groups of 6 friends that you can take with you.

4. The "AND" tells us that we need to use the Fundamental Counting Principle in this question. We need to multiply the number of ways you can pick 3 boys with the number of ways you can pick 3 girls. We have 4 boys in total to choose from. We need to choose exactly 3. We have 6 girls in total to choose from. We need to choose exactly 3. Using the Fundamental Counting Principle and the Combinations Formula

we want to solve:

$$\begin{aligned}\text{total number of ways} &= {}_4C_3 \times {}_6C_3 \\ &= \frac{4!}{3!(4-3)!} \times \frac{6!}{3!(6-3)!} \\ &= \frac{4!}{3!(1)} \times \frac{6!}{3!(3!)} \\ &= 4 \times \frac{6 \times 5 \times 4}{3!} \\ &= 80\end{aligned}$$

Thus there are 80 different ways you can pick 3 boys and 3 girls to can take with you to the movies.

Pascal's Triangle

In the 16^{th} century, **Pascal's Triangle** was named after the French mathematician Blaise Pascal because of his work but interestingly enough, Pascal was definitely not the first to arrange these numbers into a triangle. It was worked on by Jia Xian in the 11^{th} century in China, and then it was popularized in the 13^{th} century by Chinese mathematician, Yang Hui and became known as *Yang Hui's Triangle*. Yet again, even before Yang Hui, it was discussed and known in the 11^{th} century as the *Khayyam Triangle* in Iran and was named after the Persian mathematician Omar Khayyam.

Isn't cool how people are able to study and discover mathematics from different parts of the world?

For this lesson, we will call it *Pascal's Triangle*. The triangle is built as follows:

$$
\begin{array}{lcccccccccccc}
\text{row } 0 & \longrightarrow & & & & & 1 & & & & & & \\
\text{row } 1 & \longrightarrow & & & & 1 & & 1 & & & & & \\
\text{row } 2 & \longrightarrow & & & 1 & & 2 & & 1 & & & & \\
\text{row } 3 & \longrightarrow & & 1 & & 3 & & 3 & & 1 & & & \\
\text{row } 4 & \longrightarrow & 1 & & 4 & & 6 & & 4 & & 1 & & \\
\text{row } 5 & \rightarrow & 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & \text{entry } 0 & & \text{entry } 1 & & \text{entry } 2 & & \text{entry } 3 & & \text{entry } 4 & & \text{entry } 5
\end{array}
$$

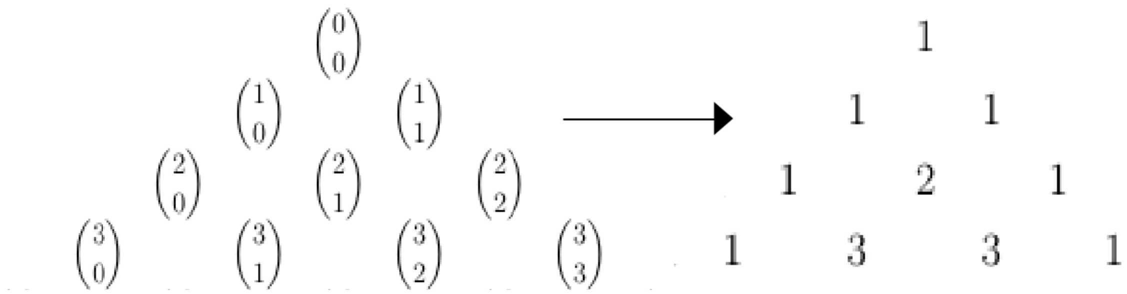
Note: In Pascal's Triangle the rows start at index 0. Similarily, the entries in each row start at index 0.

Each number in Pascal's Triangle can be written as a combination as seen below.

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

In general, the k-th entry in the n-th row can be written as $\binom{n}{k} = {}_nC_k$.

That is, for the first few entries we have:



Example: Find the number for each entry in Pascal's Triangle:

1. The 6th entry of the 8th row:

   The 6th entry of the 8th row can be written as $\binom{8}{6} = {}_8C_6$. Thus:

$$\begin{aligned}\binom{8}{6} &= \frac{8!}{6!(8-6)!} \\ &= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1 \times (2 \times 1)} \\ &= \frac{8 \times 7}{2} \\ &= 28\end{aligned}$$

   Thus the 6th entry of the 8th row is 28.

2. The 0th entry of the 3rd row:

   The 0th entry of the 3rd row can be written as $\binom{3}{0} = {}_3C_0$. Thus:

$$\begin{aligned}\binom{3}{0} &= \frac{3!}{0!(3-0)!} \\ &= \frac{3 \times 2 \times 1}{3 \times 2 \times 1} \\ &= 1\end{aligned}$$

   Thus the 0th entry of the 3rd row is 1.

3. The 12th entry of the 10th row:

   We cannot find this because the 10th row only has entries from 0 to 10. It does not have a 12th entry!

Problem Set

1. State if the question is a Fundamental Counting Principle problem, a basic Permutations problem (learnt in the previous lesson), a Permutation with Repeats problem, or a Combinations problem:

   (a) Calvin Klein has 4 shirts, 7 pants and 2 pairs of shoes. How many different outfits can Calvin Klein put together?

   Fundamental Counting Principle problem

   (b) A student club with 10 members wishes to select a president, a secretary, and a treasurer from its membership. No member may be selected for more than 1 position. In how many ways can this be done?

   Basic Permutations problem

   (c) How many different 6-digit PIN codes are there using digits from 0 to 9?

   Fundamental Counting Principle problem/Basic Permutations problem

   (d) How many different ways can the letters in the word CALCULATOR be arranged?

   Permutation with Repeats problem

   (e) How many ways can Jenny the Jeweller make a keychain with 20 distinct beads if she has 30 distinct beads? Note: The keychain is a straight piece of string.

   Basic Permutations problem

2. Christmas is coming up and Emily wants to plan out how she will decorate her fireplace mantel. Emily has 12 ornaments in total. She has:

   - 3 red stockings
   - 2 orange ball ornaments
   - 4 green ball ornaments
   - 2 angel ornaments
   - 1 blue ball ornament

   How many **different** ways can Emily arrange her ornaments on the mantel if she wants to use all 12 ornaments?

   First let's take a look at how many ornaments we have in total. How many ornaments are repeated?

   - We have 12 ornaments in total.

- We have 3 repeats of the red stockings.
- We have 2 repeats of the orange ball ornaments.
- We have 4 repeats of the green ball ornaments.
- We have 2 repeats of the angel ornaments.

Since we have 12 ornaments in total and we want to arrange all 12 ornaments, with repeats, we know this is a Permutation with Repeats problem! Here $n = 12$, $r_1 = 3$, $r_2 = 2$, $r_3 = 4$, and $r_4 = 2$. Thus using the Permutations with Repeats formula:

$$\begin{aligned}\text{total number of different arrangements} &= \frac{12!}{(3!)(2!)(4!)(2!)}\\ &= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{(3 \times 2 \times 1)(2)(2)(4!)}\\ &= \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5}{6(2)(2)}\\ &= 7560\end{aligned}$$

Therefore there are 7560 different ways can Emily arrange her ornaments on the mantel.

3. Luc, Ryan, Alysha and Vince liked Frozen 2 so much that they are going to see it a second time, and this time they're bringing Tim. When the friends go to the theatre they all sit in a row of 5 seats. Ryan and Tim want to sit together. How many different seating arrangements of the five friends are possible in a row with 5 seats?

   To solve this question we need to consider the condition that Ryan and Tim want to sit together. Since Ryan and Tim need to stay together, we can think of them as one person and then multiply by the number of ways both of them can be arranged. You can picture Ryan and Tim as the same person and wherever they are sitting, we know we can simply swap them and get another seating arrangement. That is **Ryan-Tim** can also give **Tim-Ryan**. That is, for every seating arrangement we have another one with Ryan and Tim swapped, so we need to multiply the total number of ways by 2. To get the total number of ways we count 4 people in total since we think of Ryan and Tim as 1 person. We know ordering 4 people without repetition is simply 4!. Thus:

$$\text{total number of ways} = 4! \times (2) = 4 \times 3 \times 2 \times 1 \times (2) = 48$$

4. How many words can you make rearranging the letters of the following words:

(a) MATHEMATICS

This is a Permutation with Repeats problem. Here $n = 11$, $r_1 = 2$ since we have 2 repeats of the letter 'M', $r_2 = 2$ since we have 2 repeats of the letter 'A', and $r_3 = 2$ since we have 2 repeats of the letter 'T'. Thus using the Permutations with Repeats formula:

$$\begin{aligned}\text{total number of different arrangements} &= \frac{11!}{(2!)(2!)(2!)} \\ &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2!}{(2)(2)(2!)} \\ &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3}{(2)(2)} \\ &= 4,989,600\end{aligned}$$

Therefore there are 4,989,600 different words arranged using the letters of MATHEMATICS.

(b) MISSISSIPPI

This is a Permutation with Repeats problem. Here $n = 11$, $r_1 = 4$ since we have 4 repeats of the letter 'I', $r_2 = 4$ since we have 4 repeats of the letter 'S', and $r_3 = 2$ since we have 2 repeats of the letter 'P'. Thus using the Permutations with Repeats formula:

$$\begin{aligned}\text{total number of different arrangements} &= \frac{11!}{(4!)(4!)(2!)} \\ &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4!}{(4!)(4!)(2!)} \\ &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5}{(4 \times 3 \times 2 \times 1)(2)} \\ &= 34650\end{aligned}$$

Therefore there are 34650 different words arranged using the letters of MISSISSIPPI.

5. Suppose a lottery ticket can have 5-digits from 0 to 9 on it with no repeating digits.

   (a) How many different possibilites of the winning 5-digits are there if the order of the digits don't matter?

   Since the order of the digits don't matter and we can't repeat digits, this is a Combinations problem! We need to calcuate the number of different 5-digit combinations using the digits 0 to 9. This means $n = 10$, since there are 10 digits to choose from (0,1,2,...,9) and we need to choose 5 digits so $k = 5$. Thus using the Combinations formula:

$$\begin{aligned}\text{total number of 5-digit numbers} = {}_{10}C_5 &= \frac{10!}{5!(10-5)!} \\ &= \frac{10!}{5!5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6}{5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2 \times 1} \\ &= 252\end{aligned}$$

(b) How many different possibilites of the winning 5-digits are there if the order of the digits matter?

Since the order of the digits matter and we can't repeat digits, this is a Permutations problem! We need to calcuate the number of different 5-digit arrangements using the digits 0 to 9. This means $n = 10$, since there are 10 digits from (0,1,2,...,9) and we need to arrange 5 digits so $k = 5$. Thus using the Permutations formula:

$$\begin{aligned}\text{total number of 5-digit numbers} = {}_{10}P_5 &= \frac{10!}{(10-5)!} \\ &= \frac{10!}{5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!} \\ &= 10 \times 9 \times 8 \times 7 \times 6 \\ &= 30,240\end{aligned}$$

6. Florean Fortescue's Ice Cream Parlour has 53 flavours of ice cream. How many different types of sundaes could Harry get if he wanted:

(a) 1 scoop?

Intuitivley, we know there are 53 different sundaes possible with 1 scoop. Mathematically, ${}_{53}C_1 = {}_{53}P_1 = 53$.

(b) 2 differently flavoured scoops?

Order does not matter and there is no repetition (since you want different scoops) so this is a combination problem.

$${}_{53}C_2 = \frac{53!}{2!(53-2)!} = \frac{53!}{2!51!} = 1378$$

(c) 5 differently flavoured scoops?

$${}_{53}C_5 = \frac{53!}{5!(53-5)!} = \frac{53!}{5!48!} = 2,869,685$$

7. At a cafeteria, a student is allowed to pick 4 items from the following list (one of each): pop, juice, milk, water, burger, hotdog, vegetable soup, banana, orange, apple pie.

   (a) Does order matter?

   Consider the choice (pop, juice, hotdog, banana). Is it different than the choice (pop, banana, juice)? No! - The students' meal will be the exact same. So order does not matter.

   (b) Is repetition allowed?

   No! - Repetition is not allowed since the question states we can only pick one of each

   (c) How many ways can a student have a 4 piece meal?

   Since order does not matter and repeptition is not allowed, this is a Combinations problem. In this case, we have 10 items to choose from and we need to choose 4. Then $n = 10$, and $k = 4$. Thus using the Combinations formula:

$$\begin{aligned}
\text{total number of 4 piece meals} = {}_{10}C_4 &= \frac{10!}{4!(10-4)!} \\
&= \frac{10!}{4!6!} \\
&= \frac{10 \times 9 \times 8 \times 7 \times 6!}{4!6!} \\
&= \frac{10 \times 9 \times 8 \times 7}{4!} \\
&= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \\
&= 210
\end{aligned}$$

(d) How many ways can a student have a 4 piece meal if they need to take exactly one drink?

This question follows the same idea as the above question except the student has to choose exactly one drink. Since he needs to choose one drink and there are 4 drinks to choose from, this is simply ${}_4C_1$. Now the student still needs to choose 3 items to make a 4 piece meal. We can't choose any drinks anymore so there are 6 items to choose from (all items without the drink options). This is ${}_6C_3$. This is another way of thinking about the Fundamental Counting Principle. The students needs to choose one drink AND 3 other items to make his 4 piece meal. Thus using the FCP formula and the Combinations formula we have:

$$\begin{aligned}
\text{total number of 4 piece meals} = {}_4C_1 \times {}_6C_3 &= \frac{4!}{1!(4-1)!} \times \frac{6!}{3!(6-3)!} \\
&= \frac{4!}{3!} \times \frac{6!}{3!(3)!} \\
&= \frac{4 \times 3!}{3!} \times \frac{6 \times 4 \times 3!}{3!} \\
&= 4 \times (6 \times 4) \\
&= 96
\end{aligned}$$

8. Bathilda Bagshot bought a bag of Bertie Bott's Beans. There are 21 bad tasting beans and 8 good tasting beans (all the beans are different). Bathilda wants to eat 4 beans.

(a) How many ways can Bathilda pick her beans?

Order does not matter and there is no repetition so this is a Combinations problem. There are $21 + 8 = 29$ beans in total and she needs to pick 4. Thus $n = 29$ and $k = 4$. Then:

$$\text{total number of ways} = {}_{29}C_4 = \frac{29!}{4!(29-4)!} = \frac{29!}{4!25!} = 23,751$$

(b) How many ways can Bathilda pick all good beans?

There are 8 good beans in total and she needs to pick 4.

$$\text{total number of ways} = {}_8C_4 = \frac{8!}{4!(8-4)!} = \frac{8!}{4!4!} = 70$$

(c) How many ways can Bathilda pick all bad beans?

There are 21 bad beans in total and she needs to pick 4.

$$\text{total number of ways} = {}_{21}C_4 = \frac{21!}{4!(21-4)!} = \frac{21!}{4!17!} = 5985$$

(d) How many ways can Bathilda pick 2 good beans AND 2 bad beans?

The "and" tells us that we need to use the Fundamental Counting Principle in this question. We need to multiply the number of ways Bathilda can pick 2 good beans with the number of ways Bathilda can pick 2 bad beans.

$${}_8C_2 \times {}_{21}C_2 = \frac{8!}{2!(8-2)!} \times \frac{21!}{2!(21-2)!} = \frac{8!}{2!(6)!} \times \frac{21!}{2!(19)!} = 28 \times 210 = 5880$$

(e) ** How many ways can Bathilda have more good beans than bad?

There are only two scenarios in which Bathilda has more good beans than bad beans: when she has 3 good beans and 1 bad bean OR when she has 4 good beans and 0 bad beans. Since we can't have both of these situtations at the same time, we can't use the Fundamental Counting Principle (we can't have 3 good beans and 1 bad bean AND 4 good beans and 0 bad beans). To count the total number of ways to make choice A OR choice B, we have to add the number of ways to make choice A plus the number of ways to make choice B. Thus, we have to add the number of ways for each situation. Following the same logic as the previous parts:

$$\begin{aligned}
(\text{3 good and 1 bad}) + (\text{all good}) &= ({}_8C_3 \times {}_{21}C_1) + {}_8C_4 \\
&= (\frac{8!}{3!(8-3)!} \times \frac{21!}{1!(21-1)!}) + 70 \\
&= (\frac{8!}{3!(85)!} \times \frac{21!}{1!(20)!}) + 70 \\
&= (56 \times 21) + 70 \\
&= 1246
\end{aligned}$$

9. A school has 380 female students and 120 male students. They must create a 5-person student council.

(a) In how many ways can they do this if there must be 4 boys and 1 girl?

The "and" tells us that we need to use the Fundamental Counting Principle in this question. We need to multiply the number of ways the council can pick 4 boys with the number of ways the council can pick 1 girl.

$$
\begin{aligned}
{}_{120}C_4 \times {}_{380}C_1 &= \frac{120!}{4!(120-4)!} \times \frac{380!}{1!(380-1)!} \\
&= \frac{120!}{4!(116)!} \times \frac{380!}{1!(379)!} \\
&= \frac{120 \times 119 \times 117 \times 116!}{4!(116)!} \times \frac{380 \times 379!}{379!} \\
&= \frac{120 \times 119 \times 117}{4!} \times (380) \\
&= 3121536600
\end{aligned}
$$

(b) ** In how many ways can they do this if there must be more girls than boys?

There are three scenarios in which the council has more good girls than boys: when there are 5 girls and 0 boys, when there are 4 girls and 1 boy, and when there are 3 girls and 2 boys. Since we can't have all of these situtations at the same time, we can't use the Fundamental Counting Principle. Thus, we have to ADD the number of ways for each situation. We get:

$$
\begin{aligned}
&= (\text{5 girls and 0 boys}) + (\text{4 girls and 1 boy}) + (\text{3 girls and 2 boys}) \\
&= ({}_{380}C_5) + ({}_{380}C_4 \times {}_{120}C_1) + ({}_{380}C_3 \times {}_{120}C_2) \\
&= (\frac{380!}{5!(380-5)!}) + (\frac{380!}{4!(380-4)!} \times \frac{120!}{1!(120-1)!}) \\
&+ (\frac{380!}{3!(380-3)!} \times \frac{120!}{2!(120-2)!}) \\
&= (\frac{380!}{5!(375)!}) + (\frac{380!}{4!(376)!} \times \frac{120!}{(119)!}) + (\frac{380!}{3!(377)!} \times \frac{120!}{2!(118)!}) \\
&= (\frac{380 \times 379 \times 378 \times 377 \times 376 \times 375!}{5!(375)!}) \\
&+ (\frac{380 \times 379 \times 378 \times 377 \times 376!}{4!(376)!} \times \frac{120 \times 119!}{(119)!}) \\
&+ (\frac{380 \times 379 \times 378 \times 377!}{3!(377)!} \times \frac{120 \times 119 \times 118!}{2!(118)!}) \\
&= (\frac{380 \times 379 \times 378 \times 377 \times 376}{5!}) + (\frac{380 \times 379 \times 378 \times 377}{4!} \times 120) \\
&+ (\frac{380 \times 379 \times 378}{3!} \times \frac{120 \times 119}{2!}) \\
&= 231,709,284,576
\end{aligned}
$$

10. Complete the following rows of Pascal's Triangle. (**Hint:** You can do this by writing out the entire triangle or use $\binom{n}{k}$ to find each entry).

    (a) Complete the 6th row.

    1, 6, 15, 20, 15, 6, 1

    (b) Complete the 9th row.

    1, 9, 36, 84, 126, 126, 84, 36, 9, 1

11. Find the missing number in this row. (**Hint:** Looking at the numbers is useful, but how many entries are there?)

1 _ 78 186 715 1287 1716 1716 1287 715 186 78 _ 1

1 13 78 186 715 1287 1716 1716 1287 715 186 78 13 1