Math Circles - Grade 11/12

February 17 - 23, 2021

Some problems that I like - Solution

1. Suppose that x is a real number such that $x + \frac{2}{x} = 7$. Determine $x^2 + \frac{4}{x^2}$.

   Solution: Note that $x^2 + \frac{4}{x^2}$ makes us think of $(x + \frac{2}{x})^2$. Indeed, $(x + \frac{2}{x})^2 = x^2 + 4 + \frac{4}{x^2}$, so $x^2 + \frac{4}{x^2} = (x + \frac{2}{x})^2 - 4 = 7^2 - 4 = 45$.

2. Suppose that x, y are positive real numbers that satisfy

$$\begin{aligned} xy &= \frac{1}{9} \\ x(y+1) &= \frac{7}{9} \\ y(x+1) &= \frac{5}{18} \end{aligned}$$

   Determine $(x + 1)(y + 1)$.

   Solution: Here, adding them equations doesn't look very useful. However, multiplying them will be awesome: if we multiple the 2nd and 3rd equations, we get a factor of $(x + 1)(y + 1)$ multiplied by xy, and the first equation tells us what xy is. So

$$\begin{aligned} (x+1)(y+1) &= (x+1)(y+1)\frac{xy}{xy} \\ &= [x(y+1)] \times [y(x+1)] \times \frac{1}{xy} \\ &= \frac{7}{9} \times \frac{5}{18} \times \frac{9}{1} \\ &= \frac{35}{18} \end{aligned}$$

3. Suppose that x and y satisfy the equations

$$\begin{aligned} 3\sin x + 4\cos y &= 5 \\ 4\sin y + 3\cos x &= 2 \end{aligned}$$

   Determine $\sin(x + y)$.

   Solution: It's a bit harder to see what to do here - adding them doesn't appear useful. Multiplying them is a good guess, but it also doesn't really make progress for us. Instead, let's try squaring both equations (whenever you have equations involving cosines and sines, don't be afraid to square them - stuff sometimes simplifies really nice since $\sin^2 u + \cos^2 u = 1$). Squaring each of them gives us the two new equations

$$\begin{aligned} 9\sin^2 x + 24\sin x\cos y + 16\cos^2 y &= 25 \\ 16\sin^2 y + 24\cos x\sin y + 9\cos^2 x &= 4 \end{aligned}$$

Now we add both equations: this gives us that

$$25 + 24(\sin x \cos y + \cos x \sin y) = 29$$

and the expression in parentheses is exactly $\sin(x+y)$. Therefore $\sin(x+y) = \frac{1}{6}$.

4. A list of numbers $x_1, x_2, \cdots, x_{100}$ has the following property: given any integer $1 \leq k \leq 100$, x_k is k less than the sum of the other 99 numbers. Determine $x_1 + x_2 + \cdots + x_{100}$.

   Solution: Let S denote the sum $x_1 + x_2 + \cdots + x_{100}$. Note that, for fixed k, $S - x_k$ is the sum of all the x_i's other than x_k. Thus the given condition may be rewritten as $x_k = S - x_k - k$ for all $1 \leq k \leq 100$. In other words, $x_k = \frac{1}{2}[S-k]$ for each k.

   This gives us 100 equalities, one for each k. We add all such equalities together. This gives us that

   $$S = 50S - \frac{1}{2}[1 + 2 + 3 + \cdots + 100]$$

   Since $1+2+3+\cdots+100 = \frac{100 \times 101}{2} = 5050$, we obtain that $S = 50S - 2025$, which rearranges to give $S = \frac{2025}{49}$