



CEMC Math Circles - Grade 11/12

March 31, 2020 - April 6, 2020

Comparison Machine

Your mission, should you choose to accept it, is to develop algorithms to complete tasks involving the relative order of n distinct integers. The problem is that the integers are random and unknown to you! All you know is that they are named a_1, a_2, \dots, a_n . Note that we call i the *index* of the integer a_i .

For each task, your approach must work no matter what the order of the integers is.

Some good news: A helpful machine is available. The machine knows the relative order of these integers. To use it, you enter the indices of two integers into the machine and it will tell you which of the two corresponding integers is larger. For example, if $a_4 = 5$, $a_2 = 7$ and you enter 4 and 2 into the machine, it will tell you that a_2 is larger. We name the machine M and in this case we have $M(4, 2) = 2$ and $M(2, 4) = 2$. Either of these application of M tells you that the integer with index 2 is larger than the integer with index 4.

Some bad news: For each task, there is a limit on the number of times you can use the machine. This limit applies no matter what the relative order of the n integers happens to be.

Some more good news: Your memory is perfect and you can remember (or record) the result every time you use the machine.

Example

Suppose $n = 4$ and you want to determine which of the integers, a_1 , a_2 , a_3 , or a_4 , is the largest, while limiting yourself to only 3 uses of the machine. Here is one way to do this:

1. Compute $M(1, 2)$ and record this answer as the index x .
2. Compute $M(3, 4)$ and record this answer as the index y .
3. The largest integer is the integer with index $M(x, y)$.

Important: We have written a Python computer program that will generate random integers and simulate the helpful machine. It is a lot of fun to use this interactive tool to test if your solutions are correct.

Using the Tool

The tool works by repeatedly asking you what you want to enter into the machine and then displaying the result. After the number of times you have used the machine reaches the limit, it will ask you for the index of the largest integer in the list. It will then tell you whether or not you are correct.

You do not need to know anything about Python in order to use the tool.

Getting the correct answer for a few lists does not mean that you have a correct algorithm for the task.

Your algorithm has to work for any choice of integers, regardless of their order. The more you test your algorithm, the more evidence you have that it is correct. After testing out your algorithm using the tool, try to explain why your algorithm will work on all possible lists.

Here are instructions for using the tool:

1. For the first three tasks, open [this webpage](#) in one tab of your internet browser. For the last three tasks, open [this webpage](#). Either way, you should see Python code.
2. Open [this free online Python interpreter](#) in another tab. You should see a middle panel labelled *main.py*.
3. Copy the code and paste it into the middle panel of the interpreter.
4. Hit *run*. You will interact with the tool using the right black panel, and you might want to widen this panel.
5. After completing a test, or if you encounter an error, you can hit *run* to begin another test. If you want to start over during a test, you can hit *stop* and then *run*.

The Tasks

Each task below outlines how many integers, n , there will be in the list, what you are attempting to answer about the list, and a limit on how many times you can ask the machine for help.

n	Task	Limit
7	Determine the largest integer.	6
8	Determine both the largest integer and the smallest integer.	10
8	Determine the second largest integer.	9
4	List the integers from smallest to largest.	5
9	You are given the additional information that <ul style="list-style-type: none">• $a_1 < a_2$ and,• $a_3 < a_4 < a_5 < a_6 < a_7 < a_8 < a_9$. List all 9 integers from smallest to largest.	6
5	Determine the median integer.	6