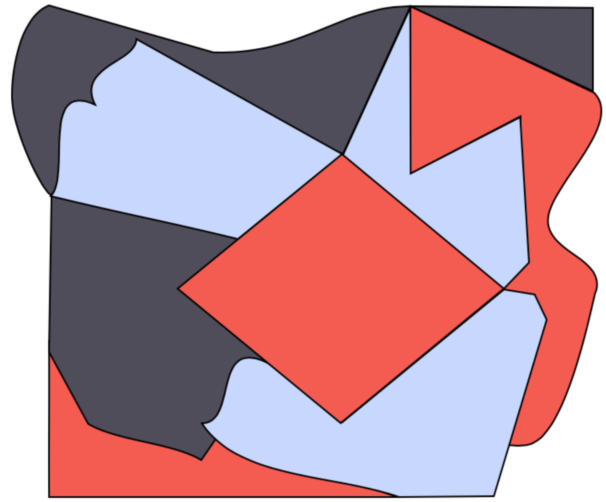
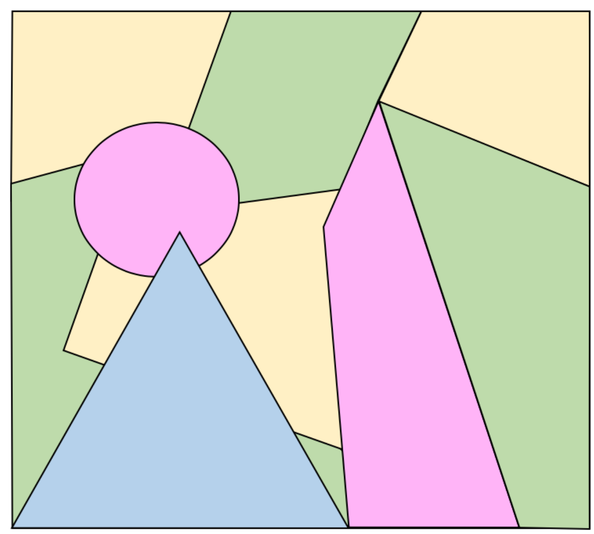
CEMC Math Circles - Grade 9/10

Wednesday, March 3, 2021
Colouring Fun with Graphs - Solution

1(a) The colouring to the right shows one way this figure can be coloured using three colours. Is there a valid colouring that uses fewer colours? Consider the region in the bottom left corner that is coloured red. This region has exactly two neighbours and these two neighbours are also neighbours of each other. Thus, in a valid colouring, these three regions must all be given different colours. That is, at least three colours are required. Since at least three colours are required and we have found a colouring that uses exactly three colours, the minimum number of colours required to colour this figure is three.



1(b) The colouring to the right shows one way this figure can be coloured using four colours. Is there a valid colouring that uses fewer colours? Consider the following four regions: the circle coloured pink along with the green region to its left, the yellow region to its bottom-left, and the blue region below it. Notice that every pair of regions in this group of four are neighbours. This means that all four of these regions must be given a different colour in a valid colouring. That is, at least four colours are required.
Since at least four colours are required and we have found a colouring that uses exactly four colours, the minimum number of colours required to colour this figure is four.



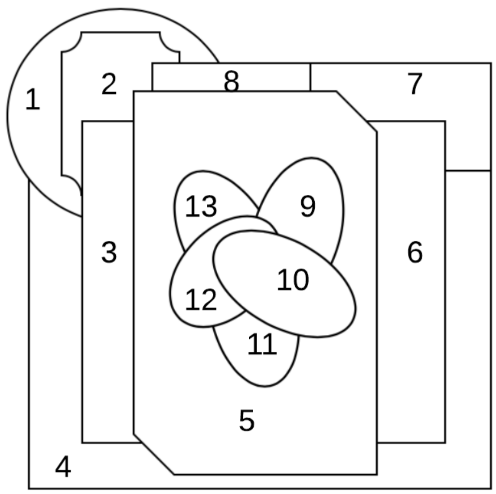
Graph colouring is the assignment of labels, in this case colours, to each vertex of a graph, `G`, such that no adjacent vertices (i.e. connected by an edge) share the same colour. The goal is to colour `G` with the minimum number of colours possible. This minimum number is known as the *chromatic number* of `G`.

There are many applications of graph colouring in the real world. Here are a few examples:

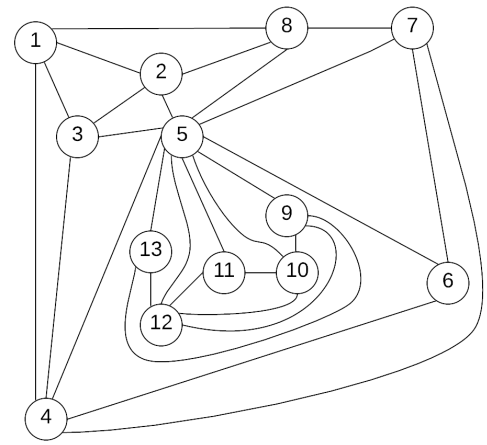
- Scheduling: classes, exams, meetings, sports, flights
- Creating and solving Sudoku puzzles
- Internet bandwidth allocation

2(a) We first work through the activity for the first figure.

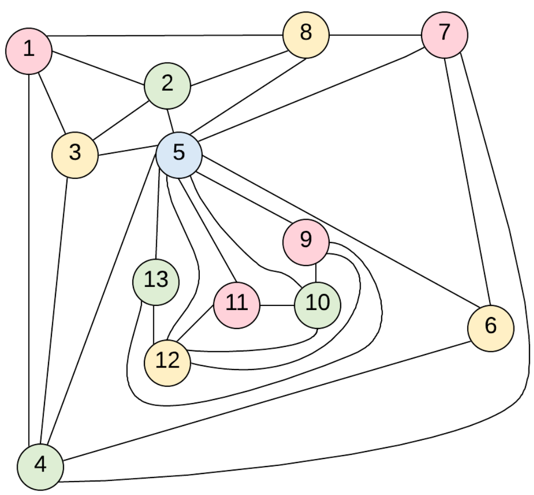
i. Label the regions



ii. Translate into a graph

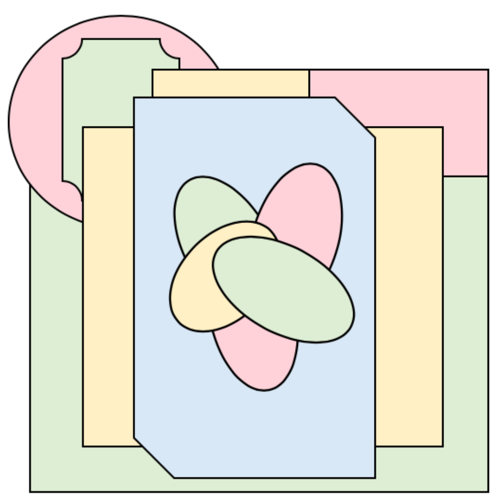


iii. Colour the graph



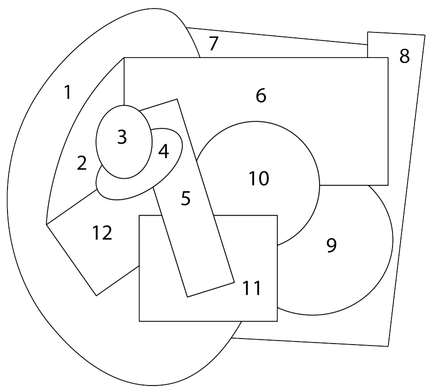
The colouring to the left shows one way this graph can be coloured using four colours. Is there a valid colouring that uses fewer colours? Notice that the group of vertices labelled 5, 10, 11, and 12 has the property that all pairs of vertices from this group are adjacent. Therefore, vertices 5, 10, 11, and 12 all require a different colour in a valid colouring, so at least four colours are required. Since at least four colours are required and we have found a colouring that uses exactly four colours, the minimum number of colours required to colour this graph is four. That is, the *chromatic number* of this graph is 4.

Finally, we use translate the colouring of the graph to a colouring of the figure:

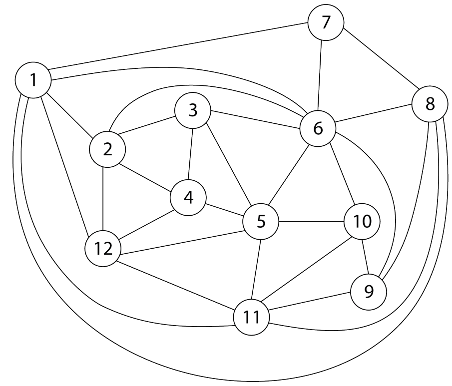


Next we work through the activity for the second figure.

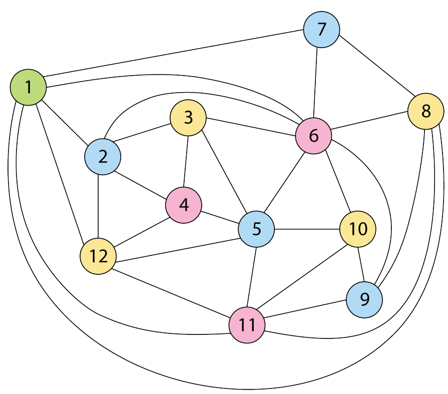
i. Label the regions



ii. Translate into a graph



iii. Colour the graph

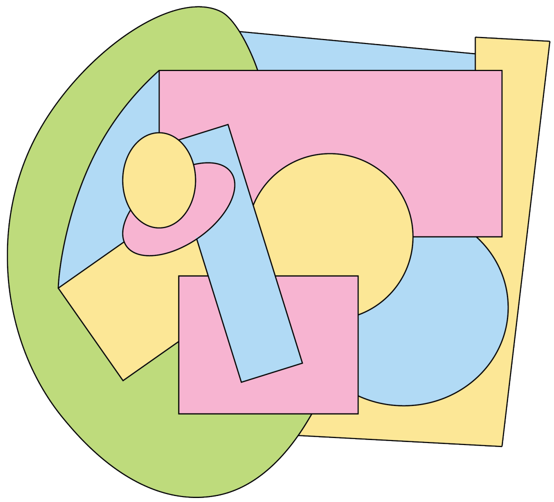


The colouring above shows one way this graph can be coloured using four colours. Is there a valid colouring that uses fewer colours? You can spend a bit of time trying to find a group of four vertices for which each pair of vertices is adjacent, but how do you proceed if you cannot find a group like this? (*Note that vertices 1, 6, 7, and 8 satisfy this property, but we will proceed as though we have not spotted these.*)

Let's try to colour this graph using only three colours: pink, yellow, and blue. Consider the vertex labelled 4 and assign this vertex a colour, say pink. (Note that exactly what colour we choose is not important here.) The vertex 4 is adjacent to the vertices labelled 2, 3, 5, and 12 and so these vertices cannot be coloured pink. Looking at the connections between these four vertices, convince yourself that since only two colours remain, vertices 2 and 5 must be coloured with the same colour, say blue, and vertices 3 and 12 must be coloured with the other colour, yellow. Now consider the vertex numbered 11, which is adjacent to vertex 5 (coloured blue) and vertex 12 (coloured yellow), and so must be coloured pink. Finally, consider the vertex numbered 1. This vertex is adjacent to vertex 2 (coloured blue), vertex 12 (coloured yellow), and vertex 11 (coloured pink). In order to produce a valid colouring, vertex 1 must be assigned a colour, but that colour cannot be pink, yellow, or blue! This means we cannot possibly finish our colouring without adding a fourth colour.

Since at least four colours are required and we have found a colouring that uses exactly four colours, the minimum number of colours required to colour this graph is four. That is, the *chromatic number* of this graph is 4.

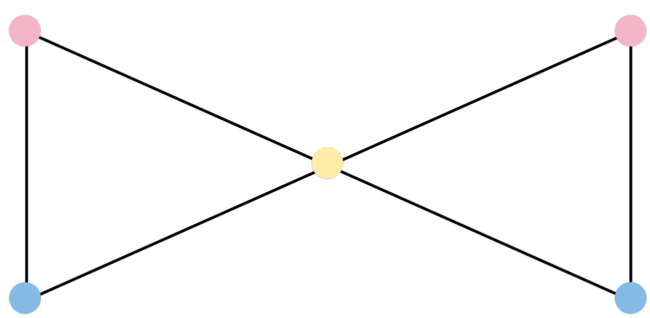
Finally, we use translate the colouring of the graph to a colouring of the figure:



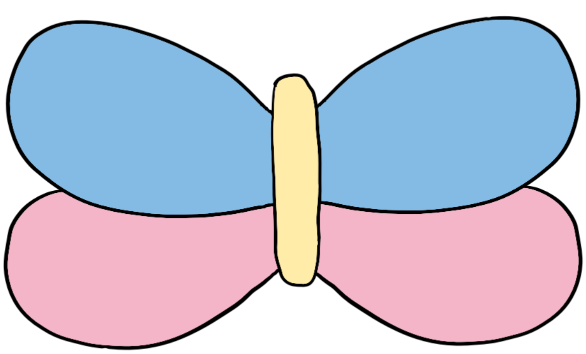
2(b) It turns out that, as hard as you try, you will not be able to create a two-dimensional figure that requires five colours in order to achieve a valid colouring. This result follows from a famous theorem in graph theory known as the *Four Colour Theorem*. The Four Colour Theorem states that any two-dimensional map, like the figures we have been colouring in this activity, always has a valid colouring with four colours. That is, it can always be coloured with four or fewer colours.

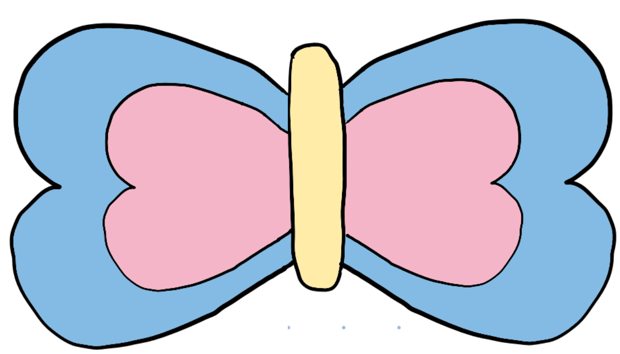
The proof of the Four Colour Theorem is not trivial. In fact, this theorem was first conjectured in 1852, but was not proven until the 1970s, and the proof required the aid of a computer!

3(a) The colouring below is one way the butterfly graph can be coloured.



3(b) Below are two possible one-dimensional figures that correspond to the butterfly graph. They are coloured as outlined by the colouring of the graph in 3(a).





3(c) See 3(b) solution above.

3(d) We can apply the method outlined on page 2, which takes a figure and finds its corresponding graph.

3(e) See 3(b) solution above.