



CEMC Math Circles - Grade 9/10

Wednesday, March 31, 2021

More Counting

In some areas of mathematics, we study things called *permutations*. A permutation of a collection of objects is an arrangement of the objects in some order.

For example, consider the integers 1, 2, and 3. There are six different ways to arrange these three objects, in some order, and so there are six permutations of these objects. The permutations are given below:

(1, 2, 3) (1, 3, 2) (3, 1, 2) (3, 2, 1) (2, 1, 3) (2, 3, 1)

In the questions below, we will work with permutations of consecutive integers, and we will think about a particular type of permutation which we will call a VALROBSAR permutation.

A permutation will be called a **VALROBSAR** permutation if *no* integer in the permutation has two neighbours that both are less than it.

Two integers in the permutation are neighbours if they appear directly beside each other.

From our example above, the permutations (1, 3, 2) and (2, 3, 1) are *not* VALROBSAR permutations. This is because, in each of these permutations, the integer 3 has a smaller integer immediately to its left and immediately to its right. That is, the integer 3 has two neighbours that are both less than 3.

The other four permutations shown above *are* VALROBSAR permutations. For example, let's look at the permutation (3, 1, 2). The integer 3 has only one neighbour and so does not have two neighbours less than 3. The integer 2 also has only one neighbour and so does not have two neighbours less than 2. The integer 1 has two neighbours but they are both greater than 1. As a second example, let's look at (3, 2, 1). The integers 3 and 1 each have only one neighbour and so do not have two neighbours less than themselves, and the integer 2 has two neighbours but only one of them is less than 2. You should work through the remaining two permutations on your own to verify that they are indeed VALROBSAR permutations.

Problems:

1. List all permutations of the integers 1, 2, 3, and 4.
2. How many of the permutations of the integers 1, 2, 3, and 4 are VALROBSAR permutations?
3. How many VALROBSAR permutations are there of the integers 1, 2, 3, 4, and 5?
4. How many VALROBSAR permutations are there of the integers 1, 2, 3, 4, 5, and 6?

Extension: Can you see a pattern forming based on your work in problems 1. to 4.? Suppose that n is a positive integer satisfying $n \geq 2$ and consider the permutations of the integers 1, 2, 3, 4, \dots , n . What can you say about the number of VALROBSAR permutations of these integers?