Intermediate Math Circles Spring 2021

Contest Prep

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Introduction

Some of you may be writing the Fryer or Galois contest later this week or next.

In this session we will look at the way the questions from the Fryer and Galois differ from the Pascal, Cayley and Fermat contests that you may have written in February.

The Difference

While the Pascal, Cayley and Fermat are multiple choice contests, the Fryer and Galois contests are full solution contests. This means each contest will be marked by a person. This is similar to the Canadian Intermediate Mathematics Contest that is written in November.

For each question, there will be two types of questions.

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For each question, there will be two types of questions.

The first type is the lightbulb question and is indicated by  .
There is a box in the solution space. If you place the correct answer in the box, then you will receive full marks.
If the answer is not correct, then the marker will look at your work in the solution space to see if there are possible part marks.

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The second type is a full solution question and is indicted by .
The marker will look at your work in the solution space to determine the number of marks that the question will receive.

More Information

Another difference is the fact that the earlier parts of each question may help you in solving the latter parts.

Finally, there could be what we refer to as a *learning* question. In this type of question, the contest will introduce some mathematical concepts that you may or may not have seen before.

Let's take a look a *learning* question from a previous contest.

Divisibility

Many of you may know the following rules to decide if a number is divisible by 2 or 5.

Divisibility by 2: A number is divisible by 2 if its last digit is even.

Divisibility by 5: A number is divisible by 5 if its last digit is 0 or 5.

This first problem we will look at introduces the divisibility test to see if a number is divisible by 3.

First Problem

The *sum of digits* of 2013 is $2 + 0 + 1 + 3 = 6$. If the sum of the digits of a positive integer is divisible by 3, then the number is divisible by 3. Also, if a positive integer is divisible by 3, then the sum of its digits is divisible by 3.

(a) List all values for the digit A such that the four-digit number $51A3$ is divisible by 3.

(b) List all values for the digit B such that the four-digit number $742B$ is divisible by both 2 and 3 (that is, is divisible by 6).

(c) Find all possible pairs of digits P and Q such that the number $1234PQPQ$ is divisible by 15.

(d) Determine the number of pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

*****Take some time to try these questions.*****

First Problem Solution - part(a)

Notice that this question wants the contest writer to use the concept just taught in the preamble of the question.
So the four-digit number $51A3$ is divisible by 3 when the sum of its digits is divisible by 3.
The sum of its digits is $5 + 1 + A + 3 = 9 + A$.
The values for the digit A such that $9 + A$ is divisible by 3 are

0, 3, 6, or 9.

NOTE: Since this is a light bulb question you could just put the answer in the box provided. However, mistakes are possible, so do show some work as well.

First Problem Solution - part(b)

Notice that this question introduces how to determine if a number is divisible by 6. That is, the number is divisible by two factors of 6 that do not share a common factor other than 1.

The four-digit number $742B$ is divisible by 2 when it is even. Thus, $742B$ is divisible by 2 if B is $0, 2, 4, 6$, or 8.

First Problem Solution - part(b)

Notice that this question introduces how to determine if a number is divisible by 6. That is, the number is divisible by two factors of 6 that do not share a common factor other than 1.

The four-digit number $742B$ is divisible by 2 when it is even.
Thus, $742B$ is divisible by 2 if B is $0, 2, 4, 6$, or 8.

The four-digit number $742B$ must also be divisible by 3.
Using the new concept taught in the preamble, the number $742B$ is divisible by 3 when $7 + 4 + 2 + B = 13 + B$ is divisible by 3.
Thus, $742B$ is divisible by 3 if B is $2, 5$ or 8.
The only values of the digit B such that the four-digit number $742B$ is divisible by 6 (divisible by both 2 and 3), are $B = 2$ or $B = 8$.

NOTE: Since this is a light bulb question you could just put the answer in the box provided. However, mistakes are possible, so do show some work as well.

First Problem Solution - part(c)

We can now use the concept shown in (b). An integer is divisible by 15 when it is divisible by both 5 and 3 (since the product of 5 and 3 is 15 and since 5 and 3 have no common factor).
The integer $1234PQPQ$ is divisible by 5 when its units digit, Q, is 0 or 5.

First Problem Solution - part(c)

We can now use the concept shown in (b). An integer is divisible by 15 when it is divisible by both 5 and 3 (since the product of 5 and 3 is 15 and since 5 and 3 have no common factor).

The integer $1234PQPQ$ is divisible by 5 when its units digit, Q, is 0 or 5.

The integer $1234PQPQ$ is divisible by 3 when the sum of its digits, $1 + 2 + 3 + 4 + P + Q + P + Q = 10 + 2P + 2Q$, is divisible by 3.

First Problem Solution - part(c)

We can now use the concept shown in (b). An integer is divisible by 15 when it is divisible by both 5 and 3 (since the product of 5 and 3 is 15 and since 5 and 3 have no common factor).

The integer $1234PQPQ$ is divisible by 5 when its units digit, Q, is 0 or 5.

The integer $1234PQPQ$ is divisible by 3 when the sum of its digits, $1+2+3+4+P+Q+P+Q = 10+2P+2Q$, is divisible by 3.

We proceed by checking the two cases, $Q = 0$ and $Q = 5$.

First Problem Solution - part(c)

We can now use the concept shown in (b). An integer is divisible by 15 when it is divisible by both 5 and 3 (since the product of 5 and 3 is 15 and since 5 and 3 have no common factor).
The integer $1234PQPQ$ is divisible by 5 when its units digit, Q, is 0 or 5.
The integer $1234PQPQ$ is divisible by 3 when the sum of its digits, $1+2+3+4+P+Q+P+Q=10+2P+2Q$, is divisible by 3.
We proceed by checking the two cases, $Q=0$ and $Q=5$.

When $Q=0$,
the sum of the digits $10+2P+2Q=10+2P+2(0)=10+2P$.
Therefore, when $Q=0$ the values of P for which $10+2P$ is divisible by 3 are $1, 4$ and 7. Therefore, three values of digits P and Q (written as (P,Q)) for which $1234PQPQ$ is divisible by 15 are

$$(1,0),(4,0), \text{ or } (7,0).$$

First Problem Solution - part(c)

When $Q = 5$,
the sum of the digits $10 + 2P + 2Q = 10 + 2P + 10 = 20 + 2P$.
Therefore, when $Q = 5$,
the values of P for which $20 + 2P$ is divisible by 3 are $2, 5$ and 8.

Therefore, another three values of digits P and Q (written as (P, Q)) for which $1234PQPQ$ is divisible by 15 are

$$(2, 5), (5, 5), \text{ or } (8, 5).$$

First Problem Solution - part(c)

When $Q = 5$,
the sum of the digits $10 + 2P + 2Q = 10 + 2P + 10 = 20 + 2P$.
Therefore, when $Q = 5$,
the values of P for which $20 + 2P$ is divisible by 3 are $2, 5$ and 8.

Therefore, another three values of digits P and Q (written as (P, Q)) for which $1234PQPQ$ is divisible by 15 are

$$(2, 5), (5, 5), \text{ or } (8, 5).$$

Finally, the six values of digits P and Q (written as (P, Q)) for which $1234PQPQ$ is divisible by 15 are

$$(1, 0), (4, 0), (7, 0), (2, 5), (5, 5), \text{ or } (8, 5).$$

First Problem Solution - part(d)

An integer is divisible by 12 when it is divisible by both 4 and 3 (since the product of 4 and 3 is 12 and since 4 and 3 have no common factor).

In the product $2CC \times 3D5$, $3D5$ is odd and thus cannot be divisible by 4.

Therefore, $2CC$ must be divisible by 4 and so must also be divisible by 2.

That is, C must be an even number.

We proceed by checking $C = 0, 2, 4, 6, 8$.

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Therefore, $2CC$ must be divisible by 4 and so must also be divisible by 2.

That is, C must be an even number.

We proceed by checking $C = 0, 2, 4, 6, 8$.

If $C = 0$, then 200 is divisible by 4 and so $C = 0$ is a possibility.

If $C = 2$, then 222 is not divisible by 4 and so $C = 2$ is not a possibility.

If $C = 4$, then 244 is divisible by 4 and so $C = 4$ is a possibility.

If $C = 6$, then 266 is not divisible by 4 and so $C = 6$ is not a possibility.

Finally, if $C = 8$, then 288 is divisible by 4 and so $C = 8$ is a possibility.

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We proceed by checking $C = 0, 2, 4, 6, 8$.

If $C = 0$, then 200 is divisible by 4 and so $C = 0$ is a possibility.

If $C = 2$, then 222 is not divisible by 4 and so $C = 2$ is not a possibility.

If $C = 4$, then 244 is divisible by 4 and so $C = 4$ is a possibility.

If $C = 6$, then 266 is not divisible by 4 and so $C = 6$ is not a possibility.

Finally, if $C = 8$, then 288 is divisible by 4 and so $C = 8$ is a possibility.

(NOTE: We could have used the fact that a number is divisible by 4 if the last two digits are divisible by 4.)

First Problem Solution - part(d)

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Therefore, $2CC$ must be divisible by 4 and so must also be divisible by 2.

That is, C must be an even number.

We proceed by checking $C = 0, 2, 4, 6, 8$.

If $C = 0$, then 200 is divisible by 4 and so $C = 0$ is a possibility.

If $C = 2$, then 222 is not divisible by 4 and so $C = 2$ is not a possibility.

If $C = 4$, then 244 is divisible by 4 and so $C = 4$ is a possibility.

If $C = 6$, then 266 is not divisible by 4 and so $C = 6$ is not a possibility.

Finally, if $C = 8$, then 288 is divisible by 4 and so $C = 8$ is a possibility.

(NOTE: We could have used the fact that a number is divisible by 4 if the last two digits are divisible by 4.)

We now proceed by using these three cases, $C = 0, C = 4$ and $C = 8$, to determine possible values for D.

First Problem Solution - part(d)

When $C = 0$, the product $2CC \times 3D5$ becomes $200 \times 3D5$.

The number 200 is not divisible by 3 since the sum of its digits, 2, is not divisible by 3.

First Problem Solution - part(d)

When $C = 0$, the product $2CC \times 3D5$ becomes $200 \times 3D5$.

The number 200 is not divisible by 3 since the sum of its digits, 2, is not divisible by 3.
Therefore, $3D5$ must be divisible by 3 and so $8 + D$ is divisible by 3.

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The number 200 is not divisible by 3 since the sum of its digits, 2, is not divisible by 3.

Therefore, $3D5$ must be divisible by 3 and so $8 + D$ is divisible by 3.

The possible values for D such that $8 + D$ is divisible by 3 are $1, 4$ and 7.

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When $C = 0$, the product $2CC \times 3D5$ becomes $200 \times 3D5$.

The number 200 is not divisible by 3 since the sum of its digits, 2, is not divisible by 3.
Therefore, $3D5$ must be divisible by 3 and so $8 + D$ is divisible by 3.
The possible values for D such that $8 + D$ is divisible by 3 are $1, 4$ and 7.
In this case, there are 3 possible pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

First Problem Solution - part(d)

When $C = 0$, the product $2CC \times 3D5$ becomes $200 \times 3D5$.

The number 200 is not divisible by 3 since the sum of its digits, 2, is not divisible by 3.
Therefore, $3D5$ must be divisible by 3 and so $8 + D$ is divisible by 3.
The possible values for D such that $8 + D$ is divisible by 3 are $1, 4$ and 7.
In this case, there are 3 possible pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

When $C = 4$, the product $2CC \times 3D5$ becomes $244 \times 3D5$.

The number 244 is not divisible by 3 since the sum of its digits, 10, is not divisible by 3.

First Problem Solution - part(d)

When $C = 0$, the product $2CC \times 3D5$ becomes $200 \times 3D5$.

The number 200 is not divisible by 3 since the sum of its digits, 2, is not divisible by 3.
Therefore, $3D5$ must be divisible by 3 and so $8 + D$ is divisible by 3.
The possible values for D such that $8 + D$ is divisible by 3 are $1, 4$ and 7.
In this case, there are 3 possible pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

When $C = 4$, the product $2CC \times 3D5$ becomes $244 \times 3D5$.

The number 244 is not divisible by 3 since the sum of its digits, 10, is not divisible by 3.

Therefore, $3D5$ must be divisible by 3 and as was shown in the first case above, there are 3 possible pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

First Problem Solution - part(d)

When $C = 8$, the product $2CC \times 3D5$ becomes $288 \times 3D5$.
The number 288 is divisible by 3 since the sum of its digits, 18, is divisible by 3.

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When $C = 8$, the product $2CC \times 3D5$ becomes $288 \times 3D5$.
The number 288 is divisible by 3 since the sum of its digits, 18, is divisible by 3.
Therefore, the product $288 \times 3D5$ is divisible by 3 independent of the value of D.

First Problem Solution - part(d)

When $C = 8$, the product $2CC \times 3D5$ becomes $288 \times 3D5$.

The number 288 is divisible by 3 since the sum of its digits, 18, is divisible by 3.

Therefore, the product $288 \times 3D5$ is divisible by 3 independent of the value of D.

That is, D can be any digit from 0 to 9.

First Problem Solution - part(d)

When $C = 8$, the product $2CC \times 3D5$ becomes $288 \times 3D5$.
The number 288 is divisible by 3 since the sum of its digits, 18, is divisible by 3.
Therefore, the product $288 \times 3D5$ is divisible by 3 independent of the value of D.
That is, D can be any digit from 0 to 9.
In this case, there are 10 possible pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

First Problem Solution - part(d)

When $C = 8$, the product $2CC \times 3D5$ becomes $288 \times 3D5$.
The number 288 is divisible by 3 since the sum of its digits, 18, is divisible by 3.
Therefore, the product $288 \times 3D5$ is divisible by 3 independent of the value of D.
That is, D can be any digit from 0 to 9.
In this case, there are 10 possible pairs of digits C and D for which the product $2CC \times 3D5$ is divisible by 12.

Therefore, the total number of pairs of digits C and D such that $2CC \times 3D5$ is divisible by 12 is $3 + 3 + 10 = 16$.

Second Problem

In the question below, A, B, M, N, P, Q, and R are non-zero digits.

(a) A two-digit positive integer AB equals $10A + B$. For example, $37 = 10 \times 3 + 7$. If $AB - BA = 72$, what is the positive integer AB?

(b) A two-digit positive integer MN is given. Explain why it is not possible $MN - NM = 80$.

(c) A three-digit positive integer PQR equals $100P + 10Q + R$.
If $P > R$, determine the number of possible values of $PQR - RQP$.

*****Take some time to try these questions.*****

Second Problem Solution - part(a)

The two-digit positive integers AB and BA equal $10A + B$ and $10B + A$, respectively.

Solving $AB - BA = 72$, we get

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The two-digit positive integers AB and BA equal $10A + B$ and $10B + A$, respectively.

Solving $AB - BA = 72$, we get

$$\begin{aligned}(10A + B) - (10B + A) &= 72\\ 9A - 9B &= 72\\ 9(A - B) &= 72\\ A - B &= \frac{72}{9}\\ A - B &= 8\end{aligned}$$

Therefore, $A - B = 8$.

Second Problem Solution - part(a)

The two-digit positive integers AB and BA equal $10A + B$ and $10B + A$, respectively.

Solving $AB - BA = 72$, we get

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Therefore, $A - B = 8$.

Since A and B are positive digits, the only possibility for which $A - B = 8$ occurs when $A = 9$ and $B = 1$.

Therefore, the positive integer AB is 91.

Second Problem Solution - part(b)

In this question we need to show why there can be no possibility for the two digit number MN. We will also use the process that we looked at in part (a).

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In this question we need to show why there can be no possibility for the two digit number MN. We will also use the process that we looked at in part (a).

The two-digit positive integers MN and NM equal $10M + N$ and $10N + M$, respectively.
Solving $MN - NM = 80$, we get

$$\begin{aligned}(10M + N) - (10N + M) &= 80 \\ 9M - 9N &= 80 \\ 9(M - N) &= 80\end{aligned}$$

Second Problem Solution - part(b)

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Solving $MN - NM = 80$, we get

$$\begin{aligned}(10M + N) - (10N + M) &= 80\\ 9M - 9N &= 80\\ 9(M - N) &= 80\end{aligned}$$

Since M and N are positive digits, $M - N$ is an integer and so $9(M - N)$ is a multiple of 9.

Second Problem Solution - part(b)

In this question we need to show why there can be no possibility for the two digit number MN. We will also use the process that we looked at in part (a).

The two-digit positive integers MN and NM equal $10M + N$ and $10N + M$, respectively.
Solving $MN - NM = 80$, we get

$$\begin{aligned} (10M + N) - (10N + M) &= 80 \\ 9M - 9N &= 80 \\ 9(M - N) &= 80 \end{aligned}$$

Since M and N are positive digits, $M - N$ is an integer and so $9(M - N)$ is a multiple of 9.
However, 80 is not a multiple of 9 and so $9(M - N) \neq 80$.

Second Problem Solution - part(b)

In this question we need to show why there can be no possibility for the two digit number MN. We will also use the process that we looked at in part (a).

The two-digit positive integers MN and NM equal $10M + N$ and $10N + M$, respectively.
Solving $MN - NM = 80$, we get

$$\begin{aligned}(10M + N) - (10N + M) &= 80 \\ 9M - 9N &= 80 \\ 9(M - N) &= 80\end{aligned}$$

Since M and N are positive digits, $M - N$ is an integer and so $9(M - N)$ is a multiple of 9.
However, 80 is not a multiple of 9 and so $9(M - N) \neq 80$.
Therefore, it is not possible that $MN - NM = 80$.

Second Problem Solution - part(c)

The three-digit positive integers PQR and RQP equal $100P + 10Q + R$ and $100R + 10Q + P$, respectively.

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Simplifying $PQR - RQP$, we get
$(100P + 10Q + R) - (100R + 10Q + P) = 99P - 99R = 99(P - R)$.

Second Problem Solution - part(c)

The three-digit positive integers PQR and RQP equal $100P + 10Q + R$ and $100R + 10Q + P$, respectively.

Simplifying $PQR - RQP$, we get
$(100P+10Q+R)-(100R+10Q+P) = 99P - 99R = 99(P - R)$.

Since P and R are positive digits, the maximum possible value of $P - R$ is 8 (which occurs when P is as large as possible and R is as small as possible, or $P = 9$ and $R = 1$).

Second Problem Solution - part(c)

The three-digit positive integers PQR and RQP equal $100P + 10Q + R$ and $100R + 10Q + P$, respectively.

Simplifying $PQR - RQP$, we get $(100P+10Q+R)-(100R+10Q+P) = 99P-99R = 99(P-R)$.

Since P and R are positive digits, the maximum possible value of $P - R$ is 8 (which occurs when P is as large as possible and R is as small as possible, or $P = 9$ and $R = 1$).

Since $P > R$, the minimum possible value of $P - R$ is 1 (which occurs when $P = 9$ and $R = 8$, for example).

Second Problem Solution - part(c)

This means, $1 \leq P - R \leq 8$ and so there are exactly 8 possible integer values of $P - R$.

Second Problem Solution - part(c)

This means, $1 \leq P - R \leq 8$ and so there are exactly 8 possible integer values of $P - R$.

Since $PQR - RQP = 99(P - R)$ and there are exactly 8 possible values of $P - R$, there are exactly 8 possible values of $PQR - RQP$.

(We note that the value of $PQR - RQP$ does not depend on the value of the digit Q.)

Your Turn

You may now try the problem set
that has three past Fryer or Galois questions