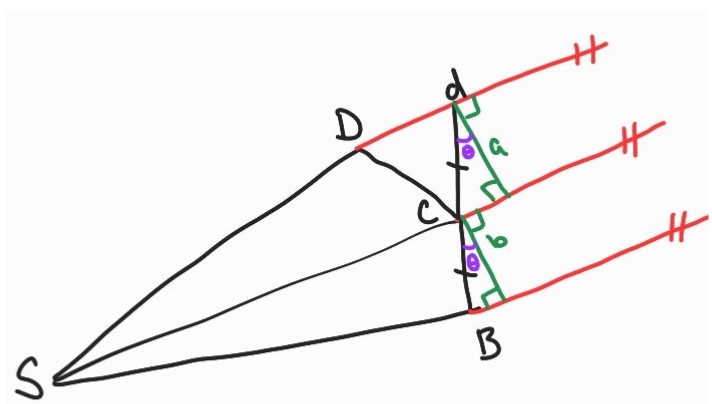
Grade 11/12 Math Circles - Fall 2021
Circles, Ellipses, and Astrophysics
Part 2: A Focus(!) on Kepler
SOLUTIONS

Exercise 1

(a) d represents the point the planet would have reached travelling in a straight line after the same time unit it took for the planet to reach C travelling in a straight line from B. Thus, the distances BC and Cd are equal, making C the midpoint of Bd.

(b) Consider the (not-to-scale) diagram of the construction below. We are trying to demonstrate that $a = b$. We have two right-angled triangles as seen in the diagram. By properties of parallel lines, we can find that the angles θ in the diagram are equal. Then, as we found that C was the midpoint of Bd and since we have right angled triangles (so each has a 90° angle), we have that the two triangles drawn are congruent (AAS). Thus, we must have that $a = b$.



(c) The common side of $\triangle SBC$ and $\triangle SCD$, SC, can be considered the base of both of the triangles. The distance between the parallel lines can be considered the heights. As these values are the same for both triangles, their areas are thus equal.

(d) $\triangle SBC$ and $\triangle SCD$ represent the area swept out by the planet after equal unit time steps. As the areas are equal, we have thus shown that we trace out equal areas in equal times.

Exercise 2

From the right-angled triangle shown in the diagram for the small-angle approximation, we have that the triangle's base is r and its height is s. However, the arclength is given by $s = r\theta$. Thus we have that the triangle's area is $A = \frac{1}{2}r(r\theta) = \frac{1}{2}\theta r^2$. Thus, we have $A \propto r^2$ as $\frac{1}{2}\theta$ is just a constant.

Exercise 3

(a) Rearranging $\left|\left|\frac{\Delta\vec{v}}{\Delta t}\right|\right| = \frac{\text{constant}}{r^2}$, we arrive at $||\Delta\vec{v}|| = \frac{\text{constant} \times \Delta t}{r^2}$.

(b) Substituting in $r^2 = \text{constant} \times A$, we arrive at $||\Delta\vec{v}|| = \dfrac{\text{constant} \times \Delta t}{A}$ (note here that the division of two constants just forms a new constant).

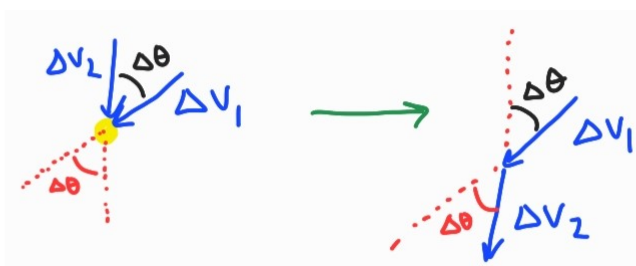
(c) Kepler's Second Law tells us that the area a planet sweeps out is proportional to the time it takes to sweep out said area (equal areas in equal times). Thus, we have that $A = \text{constant} \times \Delta t$. Substituting this in, we find that $||\Delta\vec{v}|| = \text{constant}$.

Exercise 4

(a) The length of a $\Delta\vec{v}$ vector is $||\Delta\vec{v}||$. We demonstrated in Exercise 3 that this quantity is constant, so all the $\Delta\vec{v}$ vectors thus are the same length.

(b) The length of a $\vec{v}$ vector represents how fast a planet is travelling at a given point in its orbit (longer for faster). We saw in the first lesson that a direct result of Kepler's Second Law is that a planet must be travelling faster closer to the Sun, and slower further away. So, they would be longest at perihelion, and shortest at aphelion.

(c) Consider two consecutive $\Delta\vec{v}$ vectors, and taking the note that each points directly towards the Sun. The angle between the tips of these vectors is $\Delta\theta$, as demonstrated in the first half othe diagram below. By opposite angle properties, we label another $\Delta\theta$ on the diagram. Then, sliding the second $\Delta\vec{v}$ vector down into tail-to-tip position required by a velocity diagram (the second half of the diagram), we can see that angle we labelled before matches the location we are trying to show should be $\Delta\theta$.



Exercise 5

(a) We see from the diagram that each side is of length $||\Delta\vec{v}||$.

(b) We must have n vectors in our diagram. We get one vector each time the planet sweeps out an angle of $\Delta\theta$. A full planet orbit is 360°, so we must have $n = \frac{360°}{\Delta\theta}$,

(c) For a regular n-gon, we have that the central angle for each slice as indicated in the question must be $\frac{360°}{n}$. Substituting in our answer for n from (b), we arrive at $\Delta\theta$.

(d) We saw from our investigation in lesson 1 that increasing the number of sides of an n-gon (and thus reducing the size of the angle $\Delta\theta$) leads us to approaching a circle.

Exercise 6

(a) The point P' is constructed in exactly the manner of the circle construction of an ellipse that is discussed in lesson 1. We proved that points constructed in such a matter lie on an ellipse in that lesson, thus P' lies on an ellipse.

(b) If p is perpendicular l', then as l' is perpendicular to l (due to the 90° rotation), p must be parallel to l.

Exercise 7

(a) Substituting $a = \frac{v^2}{r}$ into $F = ma$, we arrive at $F = \frac{mv^2}{r}$.

(b) Substituting $F = \frac{GMm}{r^2}$ into $F = \frac{mv^2}{r}$, we arrive at $\frac{GMm}{r^2} = \frac{mv^2}{r}$. Cancelling m on either side, and rearranging, we arrive at $v^2 = \frac{GM}{r}$.

(c) The circumference of the planet's circular orbit is $2\pi r$. Also, we have that time $= \frac{\text{distance}}{\text{velocity}}$. Thus, we arrive at the fact that the time for one orbit is $P = \frac{2\pi r}{v}$.

(d) Squaring both sides of the formula in (c) we have $P^2 = \frac{4\pi^2 r^2}{v^2}$. Then, substituting in from (b) we get $P^2 = \frac{4\pi r^3}{GM}$. Here, we see that $P^2 \propto r^3$ as $\frac{4\pi^2}{GM}$ is just a constant.

Exercise 8

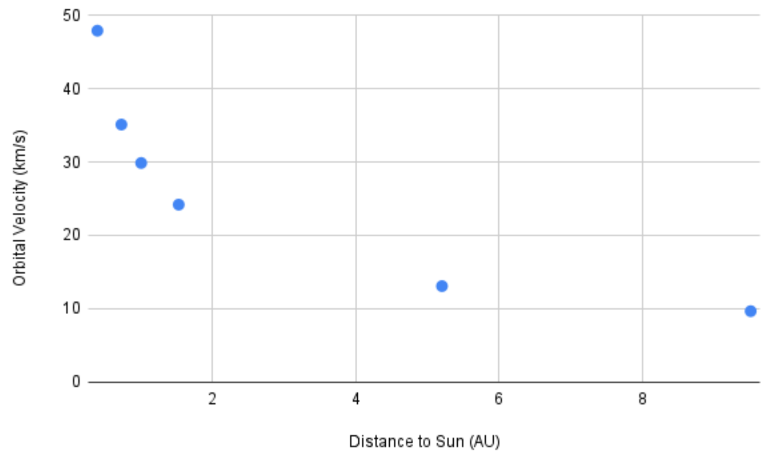
To empirically arrive at Kepler's Third Law, we are trying to show that $P^2 \propto r^3$, or in other words that $P^2 = \text{constant} \times r^3$. Thus, we want to show that $\frac{P^2}{r^3} \approx \text{constant}$. Doing this calculation, we arrive at the table below, which does indeed show that this ratio is roughly constant.

Planet	P^2/r^3
Mercury	130870.99
Venus	133042.52
Earth	133407.56
Mars	133320.02
Jupiter	133503.04
Saturn	134591.74

Exercise 9

We will use the formula $v = \sqrt{\frac{GM}{r}}$ which we obtain from Exercise 7b, and plug in $G = 6.67 \times 10^{-11}$, $M = 2 \times 10^{30}$, and for r for each planet, we take the mean distance to the sun from the table in Exercise 8 and multiply it by 1.496×10^8 in order get orbital velocities in m/s. We can then divide these answers by 1000 to get the orbital velocities in km/s. Doing this gives the table below, which can then be plotted, as below.

Planet	Mean distance to sun (AU)	Orbital velocity (km/s)
Mercury	0.389	47.88
Venus	0.724	35.09
Earth	1	29.86
Mars	1.524	24.19
Jupiter	5.20	13.10
Saturn	9.510	9.68



Exercise 10

(a) As our expectation curve drops significantly as we move to further and further radii, given that $v = \sqrt{\frac{GM}{r}}$, we expect that the radius term dominates the contained mass. Thus, we expect that the mass is mostly centrally distributed with little mass being further contained as we move radially outwards.

(b) For the curve to remain flat like this, the mass term must keep up with the radial term. Thus, we must have mass distributed throughout a galaxy, instead of just centrally. However, our observations of gas and stars, the visible matter in the universe, does not show this, so we must have invisible, dark matter throughout a galaxy as this missing mass!