Grade 6 Math Circles
October 13, 2021
Divisibility, Factors, and the GCF - Solutions

Exercise Solutions

Exercise 1

For all integers greater than 1 and less than 20, classify it as either a prime or a composite number.

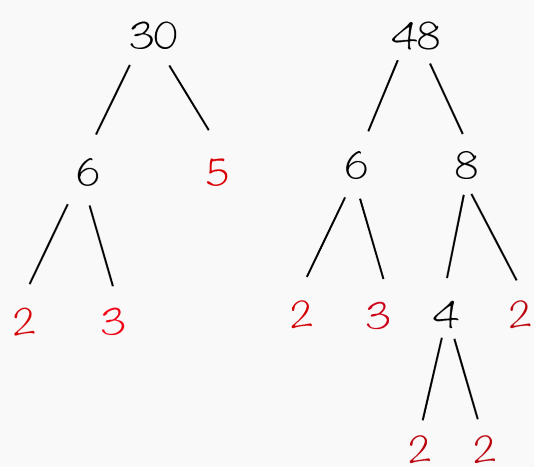
Exercise 1 Solution

The prime numbers between 1 and 20 exclusive are 2, 3, 5, 7, 11, 13, 17, and 19. That means 4, 6, 8, 9, 10, 12, 14, 15, 16, and 18 are all composite numbers.

Exercise 2

Use factor trees to find the prime factorization of 30 and 48.

Exercise 2 Solution



The prime factorization of 30 is $2 \times 3 \times 5$, and the prime factorization of 48 is $2 \times 2 \times 2 \times 2 \times 3$.

Exercise 3

How many factors does 50 have?

Exercise 3 Solution

The prime factorization of 50 is $2^1 \times 5^2$, 50 has $(1 + 1) \times (2 + 1) = 2 \times 3 = 6$ factors.

Exercise 4

Find all the factors of 50 using the factor train.

Exercise 4 Solution

1	2	5	10	25	50

Using the factor train, the 6 factors of 50 are 1, 2, 5, 10, 25, and 50.

Exercise 5

Use long division to compute $167 \div 3$, and state the quotient and remainder.

Exercise 5 Solution

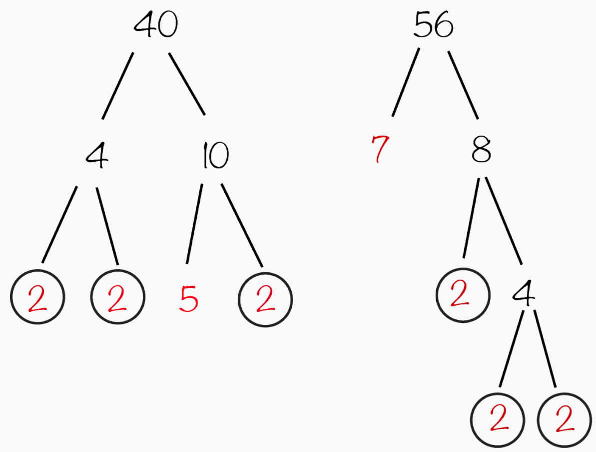
$$
\begin{array}{r}
55 \\
3\,\overline{)\,167} \\
\underline{15} \\
17 \\
\underline{15} \\
2
\end{array}
$$

Therefore, $167 \div 3 = 55$ with remainder 2.

Exercise 6

Find the GCF of 40 and 56 using factor trees. Did you obtain the same answer?

Exercise 6 Solution



When we multiply all the shared prime factors together, we get $2 \times 2 \times 2 = 8$, so the GCF of 40 and 56 is 8. We get the same answer as above, when we used the Euclidean Algorithm.

Exercise 7

Find the GCF of 36 and 54 using the Euclidean Algorithm. Did you obtain the same answer as above, when we used the factor tree method?

Exercise 7 Solution

$$\begin{array}{r} 1 \\ 36\overline{)54} \\ 36 \\ \hline 18 \end{array} \quad \rightarrow \quad \begin{array}{r} 2 \\ 18\overline{)36} \\ 36 \\ \hline 0 \end{array}$$

We see that the last non-zero remainder is 18, so the GCF of 36 and 54 is 18. We get the same answer as above, when we used the Factor Tree method.

Problem Set Solutions

1. True or False?

   (a) 16 is divisible by 4.

   (b) 15 divides 5.

   (c) The GCF of 2 and 191 is 1.

   (d) The Euclidean Algorithm always terminates.

   (e) Let x, y, and z be three positive integers. If x divides y and x also divides z, then x is called the GCF of y and z.

Solution:

(a) True

(b) False

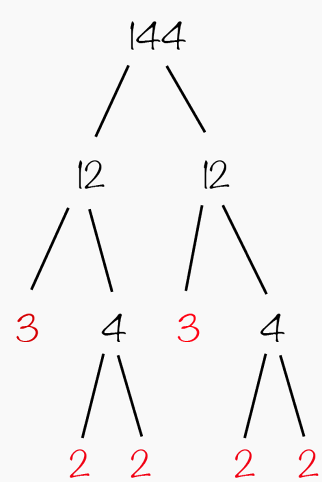
(c) True

(d) True

(e) False

2. Find the prime factorization of 144 using the factor tree method. Express your answer in exponential notation.

Solution:



The prime factorization of 144 is $2^4 \times 3^2$.

3. Janet told John that she found a positive integer with 5 factors. John told her that it's impossible, since all factors must come in pairs. Who is right?

   Solution: Janet is right. A factor can be a "pair" with itself. For example, the factors of 16 are 1, 2, 4, 8, and 16, and 4 is a "factor pair" with itself.

4. For the integers 128, 76, and 59, find the their factorizations, the number of factors they each have, and list their factors.

   Solution: The prime factorization of 128 in exponential notation is 2^7. Therefore, by the Factors Formula, the number of factors 128 has is $7 + 1 = 8$. Those factors are 1, 2, 4, 8, 16, 32, 64, and 128.
   The prime factorization of 76 in exponential notation is $2^2 \times 19$. Therefore, by the Factors Theorem, the number of factors 128 has is $(2 + 1) \times (1 + 1) = 3 \times 2 = 6$. Those factors are 1, 2, 4, 19, 38, and 76.
   Notice that 59 is a prime number. Therefore, its prime factorization is simply itself. In this case, although it is relatively trivial, we can still use the Factors Theorem. Since the prime factorization of 59 is 59^1, it has $1 + 1 = 2$ factors: 1 and 59.

5. Find the GCF of the following pairs of integers using the Euclidean algorithm.

   (a) 328 and 128
   (b) 1072 and 1184
   (c) 5 and 201
   (d) 17 and 85

Solution:

(a)

$$\begin{array}{r} 2 \\ 128\overline{)328} \\ \underline{256} \\ 72 \end{array} \rightarrow \begin{array}{r} 1 \\ 72\overline{)128} \\ \underline{72} \\ 56 \end{array} \rightarrow \begin{array}{r} 1 \\ 56\overline{)72} \\ \underline{56} \\ 16 \end{array} \rightarrow \begin{array}{r} 3 \\ 16\overline{)56} \\ \underline{48} \\ 8 \end{array} \rightarrow \begin{array}{r} 2 \\ 8\overline{)16} \\ \underline{16} \\ 0 \end{array}$$

The last non-zero remainder is 8, therefore the GCF of 328 and 128 is 8.

(b)

$$\begin{array}{r} 1 \\ 1072\overline{)1184} \\ \underline{1072} \\ 112 \end{array} \rightarrow \begin{array}{r} 9 \\ 112\overline{)1072} \\ \underline{1008} \\ 64 \end{array} \rightarrow \begin{array}{r} 1 \\ 64\overline{)112} \\ \underline{64} \\ 48 \end{array} \rightarrow \begin{array}{r} 1 \\ 48\overline{)64} \\ \underline{48} \\ 16 \end{array} \rightarrow \begin{array}{r} 3 \\ 16\overline{)48} \\ \underline{48} \\ 0 \end{array}$$

The last non-zero remainder is 16, therefore the GCF of 1072 and 1184 is 16.

(c) By inspection, you may suspect that the GCF of 5 and 201 is 1. Since 5 is prime, its only factors are 1 and itself. However, 5 does not divide 201, so 5 is not a factor of 201. This means the only common factor of 5 and 201 is 1. Let us use the Euclidean Algorithm to confirm this.

$$\begin{array}{r} 40 \\ 5\overline{)201} \\ \underline{20} \\ 01 \end{array} \rightarrow \begin{array}{r} 5 \\ 1\overline{)5} \\ \underline{5} \\ 0 \end{array}$$

The last non-zero remainder is 1, therefore the GCF of 201 and 5 is 1.

(d)

$$\begin{array}{r} 5 \\ 17\overline{)85} \\ \underline{85} \\ 0 \end{array}$$

This is a slightly special case of the algorithm. We get 0 as the remainder in our first step, which terminates the algorithm, but we do not know the value of the "previous non-zero remainder"! In this case, we can see that 17 actually **divides** 85, so 17 is a factor of 85. Since 17 is also factor of itself (this is true for all positive integers), the GCF of 17 and 85 is 17.

6. Statement: let a and b be two positive integers with $a < b$. The GCF of a and b is equal to a. Is this statement always true, sometimes true, or never true?

> *Solution*: This statement is sometimes true. Suppose a and b are two positive integers with $a < b$. The GCF of a and b is equal to a if and only if a is a factor of b. See question 5 (d) for an example. To elaborate, we will consider the following two cases.
>
> - If a is a factor of b, then a is the greatest common factor of a and b. This is because every positive integer is a factor of itself (since any positive integer is a product of 1 and itself), and a number cannot have a factor greater than itself.
> - If a is not a factor of b, then a does not divide b. Therefore a is not a common factor of a and b, which makes the statement untrue.

7. Let a and b be two positive integers. If a is a factor of b and b is also a factor of a, what is the relationship between a and b?

> *Solution*: The two integers must be equal. In order for a to be a factor of b, we must have that $a <= b$. Similarly, in order for b to be a factor of a, we must have that $b <= a$. The only way for both statements to be true is if $a = b$.

8. Let a, b, and c be three positive integers. If a is a factor of b, and b is a factor of c, is a a factor of c?

> *Solution*: Yes, a is a factor of c. Since a is a factor of b, we know that a divides b. This means that you can split b into a equal portions with nothing left over. Similarly, since b is a factor of c, you can split c into b equal portions with nothing left over. If you split c into b equal portions and then split each of the b portions into a smaller equal portions, you have successfully split c into a equal portions. Therefore, a is a factor of c.

9. A math teacher wants to split the grade 6 students into equal groups (assuming that they want more than one group). If there are 150 grade 6 students, how many ways can they make the groups?

Solution: Since the math teacher wants to make equal groups, we would like to find the factor pairs of 150 (excluding 1 and 150). You can do this using the train method.
The math teacher can split the groups in the following ways (note that you can switch the order of the factor pairs for new ways of splitting the groups!):

- 2 groups of 75
- 3 groups of 50
- 5 groups of 30
- 6 groups of 25
- 10 groups of 15
- 15 groups of 10
- 25 groups of 6
- 30 groups of 5
- 50 groups of 3
- 75 groups of 2

Therefore, they can make equal groups in 10 different ways..

10. (Challenge Question) Tracy is arranging flower vases for a dinner party. She bought 48 hyacinths, 24 tulips, and 16 azaleas to arrange into vases, and she wants every vase to look identical. If she uses all the flowers she bought, what's the greatest number of vases she can arrange? How many of each type of flowers is in any given vase? *Hint: the GCF of three positive integers a b and c is equal to the GCF of (the GCF of a and b) and c.*

Solution: Tracy wants every vase to look identical, so she must split each of the 48 hyacinths, 24 tulips, and 16 azaleas equally into x vases. Since we want x to be the largest number dividing 48, 24, and 16, x is the GCF of those three numbers.
Using the hint provided and using either the Euclidean algorithm or factor tree method, the GCF of 48 and 24 is 24. Then we find the GCF of 24 and 16, which is 8. Therefore, the greatest number of vases Tracy can arrange is 8, and each vase is filled with 6 hyacinths ($48 \div 8 = 6$), 3 tulips ($24 \div 8 = 3$), and 2 azaleas ($16 \div 8 = 2$).

11. (Challenge Question) Given that the GCF of of 252 and 105 is 21, what is the GCF of 105 and 42? You may not explicitly compute the GCF of 105 and 42. *Hint: use the fact that* $42 = 252 - (105 \times 2)$ *and the steps in the Euclidean algorithm!*

Solution: To start the Euclidean algorithm, we take the larger number to be the dividend and the smaller number to be the divisor. In each of the following steps, we take the previous divisor to be the new dividend, and take the previous remainder as the new divisor. This means that the GCF of the dividend and divisor is equal to the GCD of the divisor and remainder.
Using the hint provided, we can see that when we divide 252 by 105, the quotient is 2 and remainder is 42, and so the GCF of 105 (divisor) and 42 (remainder) is equal to the GCF of 252 (dividend) and 105 (divisor), which is 21.