Grade 7/8 Math Circles

October 13th, 2021

Solving Systems of Equations Using Matrices - Solutions

Activity Solutions

Activity 1

Use the steps above to determine a unique solution to the system of equations:

$$2x + y = 4$$
$$x - y = 5$$

Activity 1 Solution

Following step 1, we see that we can rewrite $x - y = 5$ as $x = 5 + y$. We then substitute this into the first equations, which gives:

$$\begin{aligned} 2x + y &= 4 \\ 2(5 + y) + y &= 4 \\ 2(5) + 2(y) + y &= 4 \\ 10 + 2y + y &= 4 \\ 2y + y &= 4 - 10 \\ 3y &= -6 \\ y &= -2 \end{aligned}$$

We can now substitute the values $y = -2$ into the second equation to get the value of x:

$$\begin{aligned} x &= 5 + y \\ &= 5 + (-2) \\ &= 5 - 2 \\ &= 3 \end{aligned}$$

Thus, the unique solution to the system is $x = 3$ and $y = -2$.

Activity 2

Observe the two systems of equations below:

(1)	(2)
$x + y = 1$	$x + y + z = 8$
$x - y = 2$	$2x - z = 18 - y$
$x + 2y = 3$	$3z + 2y - x = 1$

(a) Determine if each system is underdetermined, overdetermined, or balanced. Explain.

(b) Is there a unique solution to (1)? If so, state the solution. If not, how many solutions are there? Show your work.

(c) Is there a unique solution to (2)? If so, state the solution. If not, how many solutions are there? Show your work.

Activity 2 Solution

(a) For system (1), there are 2 variables and 3 equations. Thus, since there are more equations that there are variables, the system is overdetermined.
For system (2), there are 3 variables and 3 equations. Thus, since there are the same number of equations and variables, the system is balanced.

(b) From part (a), the system (1) is overdetermined, so there are either infinitely many solutions, one solution, or no solutions. To determine which of these is true for this system, we follow the four steps above.
We can rewrite the first equation to get $x = 1 - y$. We can this substitute this into the second equation to get:

$$\begin{aligned} x - y &= 2 \\ 1 - y - y &= 2 \\ 1 - 2y &= 2 \\ -2y &= 2 - 1 \\ -2y &= 1 \\ y &= -\frac{1}{2} \end{aligned}$$

We can substitute the value $y = -\frac{1}{2}$ back into the first equation to get the value of x:

$$\begin{aligned} x &= 1 - y \\ &= 1 - (-\frac{1}{2}) \\ &= 1 + \frac{1}{2} \\ &= \frac{2}{2} + \frac{1}{2} \\ &= \frac{2+1}{2} \\ &= \frac{3}{2} \end{aligned}$$

Thus, we have the solution $x = \frac{3}{2}$ and $y = -\frac{1}{2}$, which is unique for the first two equations. Now, we just have to make sure it is also a solution for the third equation. We can do this by substituting the values of x and y into the left side of the equation, and making sure the result is 3. So,

$$\begin{aligned} x + 2y &= \frac{3}{2} + 2(-\frac{1}{2}) \\ &= \frac{3}{2} + (-1) \\ &= \frac{3}{2} - 1 \\ &= \frac{3}{2} - \frac{2}{2} \\ &= \frac{3-2}{2} \\ &= \frac{1}{2} \end{aligned}$$

Since the result is $\frac{1}{2}$, and not 3, this means that $x = \frac{3}{2}$ and $y = -\frac{1}{2}$ cannot be solution to the third equation. And, since $x = \frac{3}{2}$ and $y = -\frac{1}{2}$ are unique solutions to the first two equations, there are no other solutions for the first two equations. Thus, the system (1) has no solutions.

(c) From part (a), the system (2) is balanced, so there are either infinitely many solutions, one solution, or no solutions. To determine which of these is true for this system, we follow the four steps above.

We can rewrite the third equation to get $x = 2y + 3z - 1$. We can this substitute this into the second equation to get:

$$\begin{aligned} 2x - z &= 18 - y \\ 2(2y + 3z - 1) - z &= 18 - y \\ 2(2y) + 2(3z) + 2(-1) - z &= 18 - y \\ 4y + 6z - 2 - z &= 18 - y \\ 4y + 5z + y &= 18 + 2 \\ 5y + 5z &= 20 \\ y + z &= 4 \end{aligned}$$

Once again, we can rewrite this equation as $y = 4 - z$. We can substitute this, and the equation $x = 2y + 3z - 1$, into the first equation to get:

$$\begin{aligned} x + y + z &= 8 \\ 2y + 3z - 1 + 4 - z + z &= 8 \\ 2(4 - z) + 3z - 1 + 4 - z + z &= 8 \\ 8 - 2z + 3z - 1 + 4 - z + z &= 8 \\ z + 11 &= 8 \\ z &= 8 - 11 \\ z &= -3 \end{aligned}$$

We can now substitute $z = -3$ into $y = 4 - z$ to get:

$$\begin{aligned} y &= 4 - z \\ &= 4 - (-3) \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

Finally, we substitute both $z = -3$ and $y = 7$ into $x = 2y + 3z - 1$ to get:

$$\begin{aligned} x &= 2y + 3z - 1 \\ &= 2(7) + 3(-3) - 1 \\ &= 14 + (-9) - 1 \\ &= 14 - 9 - 1 \\ &= 4 \end{aligned}$$

Thus, we have the unique solution for the system to be $x = 4$, $y = 7$ and $z = -3$.

Activity 3

For each of the four matrices above that are not in RREF, determine which of the four conditions of a matrix in RREF are not true.

Activity 3 Solution

For the first matrix, the fourth condition is not true because there is a 3 in the same column as the leading 1 in the second row.

For the second matrix, the third condition is not true because there is a row with all 0 elements above a row with a non-zero element.

For the third matrix, the first condition is not true because the first element in the second row is 2, not a leading 1.

For the fourth matrix, the second condition is not true because the leading in the second row is in a column to the left of the leading 1 in the first row.

Activity 4

Determine the RREF of $\begin{bmatrix} 2 & 6 & 4 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \end{bmatrix}$ using EROs. Show your work.

Activity 4 Solution

The RREF of the first matrix is determined by the following:

$$\begin{bmatrix} 2 & 6 & 4 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix} \begin{matrix} \xrightarrow{\frac{1}{2}\times R_1} \\ \rightarrow \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix} \begin{matrix} \rightarrow \\ \xrightarrow{R_2 - R_1} \\ \xrightarrow{R_3 - 2\times R_1} \end{matrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -1 & 3 \\ 0 & -5 & -3 \end{bmatrix} \begin{matrix} \rightarrow \\ \xrightarrow{-1\times R_2} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & -3 \\ 0 & -5 & -3 \end{bmatrix} \begin{matrix} \xrightarrow{R_1 - 3\times R_2} \\ \rightarrow \\ \xrightarrow{R_3 + 5\times R_2} \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -3 \\ 0 & 0 & -18 \end{bmatrix} \begin{matrix} \rightarrow \\ \rightarrow \\ \xrightarrow{-\frac{1}{18}\times R_3} \end{matrix} \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \xrightarrow{R_1 - 11\times R_3} \\ \xrightarrow{R_2 + 3\times R_3} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the RREF of $\begin{bmatrix} 2 & 6 & 4 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$.

The RREF of the second matrix is given by:

$$\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \end{bmatrix} \begin{matrix} \rightarrow \\ \xrightarrow{R_2 - 4\times R_1} \\ \xrightarrow{R_3 - 5\times R_1} \end{matrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 7 \\ 0 & -7 & 7 \end{bmatrix} \begin{matrix} \rightarrow \\ \xrightarrow{-\frac{1}{7}\times R_2} \\ \rightarrow \end{matrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & -7 & 7 \end{bmatrix} \begin{matrix} \xrightarrow{R_1 - 2\times R_2} \\ \rightarrow \\ \xrightarrow{R_3 + 7\times R_2} \end{matrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ is the RREF of $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \end{bmatrix}$.

Problem Set Solutions

1. Determine if the following systems of equations are underdetermined, overdetermined or balanced. Provide a brief explanation.

   (a) $7x - 20y = 4$
   $4x + 2y = 7$
   $3y - z = 100$

   (b) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{array}\right]$

   (c) $\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right]$

   (d) $y - 5x = -3$
   $10x + 7y - z = 0$

   (e) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{array}\right]$

   (f) $7 = -14x$
   $2x = -1$

   Solution:

   (a) This system has 3 variables: x, y and z; and 3 equations. Since the number of variables is equal to the number of equations, this system is balanced.

   (b) The left side of this augmented matrix has 3 columns and 2 rows. Since there are more columns than rows, this system is underdetermined.

   (c) The left side of this augmented matrix has 2 columns and 3 rows. Since there are more rows than columns, this system is overdetermined.

(d) This system has 3 variables: x, y and z; and 2 equations. Since the number of variables is greater than the number of equations, this system is underdetermined.

(e) The left side of this augmented matrix has 3 columns and 3 rows. Since there are the same number of rows and columns, this system is balanced.

(f) This system has 1 variable 1: x; and 2 equations. Since the number of equations is greater than the number of variables, this system is overdetermined.

2. Convert the following system of equations into an augmented matrix:

$$\begin{aligned} 3a + 2b - 4c + 7d &= 15 \\ a - 3b - d &= 0 \\ 4b + 4c &= 5 \end{aligned}$$

Solution: This system of equations is equivalent to the following augmented matrix:

$$\left[\begin{array}{cccc|c} 3 & 2 & -4 & 7 & 15 \\ 1 & -3 & 0 & -1 & 0 \\ 0 & 4 & 4 & 0 & 5 \end{array}\right]$$

3. Convert the following augmented matrix into a system of equations, with variables of your choosing:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 100 \\ 1 & -2 & 0 & 3 & 36 \\ 0 & -6 & 4 & -9 & -50 \\ 0 & 0 & 0 & 1 & 20 \end{array}\right]$$

Solution:
Answers will vary depending on variables used. However, the general form of the equivalent system of equations for this augmented matrix is:

$$a + b + c + d = 100$$

$$a - 2b + 3d = 36$$
$$-6b + 4c - 9d = -50$$
$$d = 20$$

4. Determine a unique solution for the following systems of equations using substitution. If there is no unique solution, state how many solutions the system has.

   (a) $18y - 2x = 6$
   $6x - 54y = 2$

   (b) $3x - y = 13$
   $3y - 2x = -4$

   (c) $4x + y = 7$
   $9y - 2x = -13$
   $2x - 9y = 13$

Solution:

(a) Using algebra, we can isolate x in the first equation by rewriting the equation as $x = 9y - 3$. We can then substitute this into the second equation, which gives:

$$\begin{aligned} 6x - 54y &= -4 \\ 6(9y - 3) - 54y &= -4 \\ 6(9y) - 6(3) - 54y &= -4 \\ 54y - 18 - 54y &= 2 \\ -18 &= 2 \text{ ?} \end{aligned}$$

Clearly this equation doesn't make any mathematical sense, but it does tell us that there are no solutions to this system of equations.

(b) Using algebra, we can rewrite the first equation to isolate y, giving $y = 3x - 13$. We can then substitute this into the second equation, which gives:

$$\begin{aligned} 3y - 2x &= -4 \\ 3(3x - 13) - 2x &= -4 \\ 3(3x) - 3(13) - 2x &= -4 \\ 9x - 39 - 2x &= -4 \\ 7x &= -4 + 39 \\ 7x &= 35 \\ x &= \frac{35}{7} \\ x &= 5 \end{aligned}$$

We can then substitute $x = 5$ into $y = 3x - 13$ to get the value of y:

$$\begin{aligned} y &= 3x - 13 \\ &= 3(5) - 13 \\ &= 15 - 13 \\ &= 2 \end{aligned}$$

Thus, we have the unique solution $x = 5$ and $y = 2$ to the system of equations.

(c) Using algebra, we can rewrite the first equation to isolate y, giving $y = 7 - 4x$. We can substitute this into the second equation, which gives:

$$\begin{aligned} 9y - 2x &= -13 \\ 9(7 - 4x) - 2x &= -13 \\ 9(7) - 9(4x) - 2x &= -13 \\ 63 - 36x - 2x &= -13 \\ -38x &= -13 - 63 \\ -38x &= -76 \\ x &= \frac{-76}{-38} \\ x &= 2 \end{aligned}$$

We can then substitute $x = 2$ into $y = 7 - 4x$ to get the value of y:

$$\begin{aligned} y &= 7 - 4x \\ &= 7 - 4(2) \\ &= 7 - 8 \\ &= -1 \end{aligned}$$

So, we have a unique solution for the first two equations, but now we need to see if $x = 2$ and $y = -1$ is also a solution for the third equation. We can determine this by substitute these values of x and y into the left side of the third equation, and seeing if the result is 13. So,

$$\begin{aligned} 2x - 9y &= 2(2) - 9(-1) \\ &= 4 - (-9) \\ &= 4 + 9 \\ &= 13 \end{aligned}$$

Thus, $x = 2$ and $y = -1$ is a unique solution to this system of equations.

5. Determine a unique solution for the following systems of equations using matrices. If there is no unique solution, state how many solutions the system has.

   (a) $-3x + 10y = 6$
   $6y - x = 2$

   (b) $-8x + 4y = 30$
   $y = 2x$

   (c) $4x + y + z = 7$
   $9y - 2x + z = -13$
   $2x - 9y - z = 13$

Solution:

(a) Our first step will be to rewrite the equations so that the variables are all in the same order on the left side, and all the constants are on the right side. This gives us:

$$-3x + 10y = 6$$
$$-x + 6y = 2$$

We can now convert this system of equations into an augmented matrix:

$$\left[\begin{array}{cc|c} -3 & 10 & 6 \\ -1 & 6 & 2 \end{array}\right]$$

Now, we have our augmented matrix, which we will transform into RREF:

$$\left[\begin{array}{cc|c} -3 & 10 & 6 \\ -1 & 6 & 2 \end{array}\right] \xrightarrow[R_2 \leftrightarrow R_1]{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} -1 & 6 & 2 \\ -3 & 10 & 6 \end{array}\right] \xrightarrow{-1 \times R_1} \left[\begin{array}{cc|c} 1 & -6 & -2 \\ -3 & 10 & 6 \end{array}\right] \xrightarrow{R_2 + 3 \times R_1}$$

$$\left[\begin{array}{cc|c} 1 & -6 & -2 \\ 0 & -8 & 0 \end{array}\right] \xrightarrow{-\frac{1}{8} \times R_2} \left[\begin{array}{cc|c} 1 & -6 & -2 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 + 6 \times R_2} \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 0 \end{array}\right]$$

Thus, we have the unique solution $x = -2$ and $y = 0$ for the system of equations.

(b) Our first step will be to rewrite the equations so that the variables are all in the same order on the left side, and all the constants are on the right side. This gives us:

$$-8x + 4y = 30$$
$$-2x + y = 0$$

We can now convert this system of equations into an augmented matrix:

$$\left[\begin{array}{cc|c} -8 & 4 & 30 \\ -2 & 1 & 0 \end{array}\right]$$

Now, we have our augmented matrix, which we will tranform into RREF:

$$\left[\begin{array}{cc|c} -8 & 4 & 30 \\ -2 & 1 & 0 \end{array}\right] \begin{array}{c}\rightarrow \\ \xrightarrow{-\frac{1}{8}\times R_1}\end{array} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{15}{4} \\ -2 & 1 & 0 \end{array}\right] \begin{array}{c}\rightarrow \\ \xrightarrow{R_2+2\times R_1}\end{array} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & -\frac{15}{4} \\ 0 & 0 & -\frac{1}{2} \end{array}\right]$$

Since the last row as all 0 elements, except for the last column, which is $-\frac{1}{2}$, we have that the system has no solutions.

(c) Our first step will be to rewrite the equations so that the variables are all in the same order on the left side, and all the constants are on the right side. This gives us:

$$\begin{aligned} 4x + y + z &= 7 \\ -2x + 9y + z &= -13 \\ 2x - 9y - z &= 13 \end{aligned}$$

We can now convert this system of equations into an augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 7 \\ -2 & 9 & 1 & -13 \\ 2 & -9 & -1 & 13 \end{array}\right]$$

Now, we have our augmented matrix, which we will tranform into RREF:

$$\left[\begin{array}{ccc|c} 4 & 1 & 1 & 7 \\ -2 & 9 & 1 & -13 \\ 2 & -9 & -1 & 13 \end{array}\right] \begin{array}{c}\xrightarrow{\frac{1}{4}\times R_1} \\ \rightarrow \\ \rightarrow\end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{1}{4} & \frac{7}{4} \\ -2 & 9 & 1 & -13 \\ 2 & -9 & -1 & 13 \end{array}\right] \begin{array}{c}\rightarrow \\ \xrightarrow{R_2+2\times R_1} \\ \xrightarrow{R_2-2\times R_1}\end{array}$$

$$\left[\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{1}{4} & \frac{7}{4} \\ 0 & \frac{19}{2} & \frac{3}{2} & -\frac{19}{2} \\ 0 & -\frac{19}{2} & -\frac{3}{2} & \frac{19}{2} \end{array}\right] \begin{array}{c}\rightarrow \\ \xrightarrow{\frac{2}{19}\times R_2} \\ \rightarrow\end{array} \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & \frac{1}{4} & \frac{7}{4} \\ 0 & 1 & \frac{3}{19} & -1 \\ 0 & -\frac{19}{2} & -\frac{3}{2} & \frac{19}{2} \end{array}\right] \begin{array}{c}\xrightarrow{R_1-\frac{1}{4}\times R_2} \\ \rightarrow \\ \xrightarrow{R_3+\frac{19}{2}\times R_2}\end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{16}{76} & 2 \\ 0 & 1 & \frac{3}{19} & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since both of the rows with leading 1s have a non-zero element in their row (exluding the last column), there are an infinite number of solutions to the system of equations.

Bonus Questions

7. At a county festival there are two types of tickets with different prices: **Adult**, which costs \$5, and **Child**, which costs \$2. On Saturday, 1000 people entered the festival and \$3350 was collected in ticket sales.

   (a) Determine the number of adults and the number of children that visited the festival on Saturday using substitution.

   (b) Determine the number of adults and the number of children that visited the festival on Saturday using matrices.

Solution:

We start by defining two variables to represent the number of adults, a, and the number of children, c. We then use the information above to create equations to represent the situation. Since there was 1000 people that entered the festival, we know that the sum of the number of adults and the number of children must be 1000. So, we have our first equation:

$$(1) \quad a + c = 1000$$

Since the adult tickets cost \$5, the total sales of adult tickets is $5a$. Similarly, the total sales of child tickets is $2c$. Thus, the sum of these two values is the total amount collected in ticket sales, which is \$3350. So, we have our second equation:

$$(2) \quad 5a + 2c = 3350$$

Thus, we have 2 variables and 2 equations, so our system is balanced.

(a) For substitution, our first step is to rewrite the first equation as $c = 1000 - a$. Now, we can substitute this into the second equation, which gives:

$$\begin{aligned} 5a + 2c &= 3350 \\ 5a + 2(1000 - a) &= 3350 \\ 5a + 2(1000) - 2(a) &= 3350 \\ 5a + 2000 - 2a &= 3350 \\ 3a &= 3350 - 2000 \\ 3a &= 1350 \\ a &= \frac{1350}{3} \\ a &= 450 \end{aligned}$$

We can substitute $a = 450$ into $c = 1000 - a$ to get the value of c:

$$\begin{aligned} c &= 1000 - a \\ &= 1000 - 450 \\ &= 550 \end{aligned}$$

Thus, we have $a = 450$ and $c = 550$, which means that there were 450 adults and 550 children at the festival on Saturday.

(b) For matrices, we can create an augmented matrix using the two equations above:

$$\left[\begin{array}{cc|c} 1 & 1 & 1000 \\ 5 & 2 & 3350 \end{array}\right]$$

Now we will solve the problem by transforming this matrix to RREF using EROs:

$$\left[\begin{array}{cc|c} 1 & 1 & 1000 \\ 5 & 2 & 3350 \end{array}\right] \xrightarrow[R_2 - 5\times R_1]{} \left[\begin{array}{cc|c} 1 & 1 & 1000 \\ 0 & -3 & -1650 \end{array}\right] \xrightarrow[-3\times R_2]{} \left[\begin{array}{cc|c} 1 & 1 & 1000 \\ 0 & 1 & 550 \end{array}\right] \xrightarrow{R_1 - R_2} \left[\begin{array}{cc|c} 1 & 0 & 450 \\ 0 & 1 & 550 \end{array}\right]$$

Thus, once again, we have $a = 450$ and $c = 550$, which means that there were 450 adults and 550 children at the festival on Saturday.

8. There are four teams competing in a sports tournament, which are simply known as Red Team,

Blue Team, Pink Team, and Orange Team. Each team has many loyal fans that attend the tournament to cheer on their favourite team. We want to figure out exactly how many fans their are each team, but we only have the following information:

- The Red Team and Orange have 8100 fans combined.
- The number of Blue Team fans multiplied by 3 is 1700 more than the number of Pink Team fans multiplied by 2.
- The Orange Team has 400 fans more than the Pink Team.
- The combined number of Blue Team and Orange team fans is 1800 more than the number of Red Team fans.
- There are 14000 fans in total at the tournament.

Using either substitution or matrices, determine how many fans there are for each team.

Solution:
We begin by defining variables to represent the number of fans for each team. We will label them as r for the Red Team, b for the Blue Team, p for the Pink Team, and o for the Orange Team. We can use the information above to create equations to represent this situation, which gives us the following five equations:

$$(1)\quad r+o=8100 \qquad (2)\quad 3b-2p=1700 \qquad (3)\quad o-p=400$$
$$(4)\quad b+o-r=1800 \qquad (5)\quad r+b+p+o=14000$$

We can then write this as a proper system of equations, with the variables ordered on the left and constants on the right, as seen below:

$$\begin{aligned} r+o&=8100\\ 3b-2p&=1700\\ -p+o&=400\\ -r+b+o&=1800\\ r+b+p+o&=14000 \end{aligned}$$

We will solve this problem using both methods.
For **substitution**, we can begin by rewriting the first equation as $r = 8100 - o$ to isolate r. We can then substitute this into the fourth equation, which gives:

$$\begin{aligned}
-r+b+o &= 1800\\
-(8100-o)+b+o &= 1800\\
-8100+o+b+o &= 1800\\
b+2o &= 1800+8100\\
b+2o &= 1800+8100\\
b+2o &= 9900\\
b &= 9900-2o
\end{aligned}$$

Then, we can rewrite the third equation as $p = o - 400$. Now we can substitute $r = 8100 - o$, $b = 9900 - 2o$ and $p = o - 400$ into the fifth equation, which gives:

$$\begin{aligned}
r+b+p+o &= 14000\\
8100-o+9900-2o+o-400+o &= 14000\\
17600-o &= 14000\\
-o &= 14000-17600\\
-o &= -3600\\
o &= 3600
\end{aligned}$$

We can use $o = 3600$ to solve for the values of the other variables, which gives:

$$r = 8100 - o = 8100 - 3600 = 4500$$
$$b = 9900 - 2(3600) = 9900 - 7200 = 2700$$
$$p = o - 400 = 3600 - 400 = 3200$$

So, we have $r = 4500$, $b = 2700$, $p = 3200$ and $o = 3600$ is a unique solution for four of the five equations. We still have to check that it is a solution for the second equation, which we will do by substituting the values on the left and seeing if the answer is 1700.

$$\begin{aligned}
3b-2p &= 3(2700)-2(3200)\\
&= 8100-6400\\
&= 1700
\end{aligned}$$

Thus, it is a unique solution for the system of equations. Hence, there are 4500 Red Team fans, 2700 Blue Team fans, 3200 Pink Team fans, and 3600 Orange Team fans.

For **matrices**, we can transform the above system of equations into the following augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 8100 \\ 0 & 3 & -2 & 0 & 1700 \\ 0 & 0 & -1 & 1 & 400 \\ -1 & 1 & 0 & 1 & 1800 \\ 1 & 1 & 1 & 1 & 14000 \end{array}\right]$$

Now, we we will solve the problem by transforming the matrix into RREF, which gives:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 8100 \\ 0 & 3 & -2 & 0 & 1700 \\ 0 & 0 & -1 & 1 & 400 \\ -1 & 1 & 0 & 1 & 1800 \\ 1 & 1 & 1 & 1 & 14000 \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \xrightarrow{R_4+R_1} \\ \xrightarrow{R_5-R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 8100 \\ 0 & 3 & -2 & 0 & 1700 \\ 0 & 0 & -1 & 1 & 400 \\ 0 & 1 & 0 & 2 & 9900 \\ 0 & 1 & 1 & 0 & 5900 \end{array}\right] \begin{array}{c} \rightarrow \\ \xrightarrow{\frac{1}{3}\times R_2} \\ \rightarrow \\ \rightarrow \\ \rightarrow \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 8100 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1700}{3} \\ 0 & 0 & -1 & 1 & 400 \\ 0 & 1 & 0 & 2 & 9900 \\ 0 & 1 & 1 & 0 & 5900 \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \xrightarrow{R_4-R_2} \\ \xrightarrow{R_5-R_2} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 8100 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1700}{3} \\ 0 & 0 & -1 & 1 & 400 \\ 0 & 0 & \frac{2}{3} & 2 & \frac{28000}{3} \\ 0 & 0 & \frac{5}{3} & 0 & \frac{16000}{3} \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \xrightarrow{-1\times R_3} \\ \rightarrow \\ \rightarrow \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 8100 \\ 0 & 1 & -\frac{2}{3} & 0 & \frac{1700}{3} \\ 0 & 0 & 1 & -1 & -400 \\ 0 & 0 & \frac{2}{3} & 2 & \frac{28000}{3} \\ 0 & 0 & \frac{5}{3} & 0 & \frac{16000}{3} \end{array}\right] \begin{array}{c} \rightarrow \\ \xrightarrow{R_2+\frac{2}{3}\times R_3} \\ \rightarrow \\ \xrightarrow{R_4-\frac{2}{3}\times R_3} \\ \xrightarrow{R_5-\frac{5}{3}\times R_3} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 8100 \\ 0 & 1 & 0 & -\frac{2}{3} & 300 \\ 0 & 0 & 1 & -1 & -400 \\ 0 & 0 & 0 & \frac{8}{3} & \frac{28800}{3} \\ 0 & 0 & 0 & \frac{5}{3} & \frac{18000}{3} \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \xrightarrow{\frac{3}{8}\times R_4} \\ \rightarrow \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 8100 \\ 0 & 1 & 0 & -\frac{2}{3} & 300 \\ 0 & 0 & 1 & -1 & -400 \\ 0 & 0 & 0 & 1 & 3600 \\ 0 & 0 & 0 & \frac{5}{3} & \frac{18000}{3} \end{array}\right] \begin{array}{c} \xrightarrow{R_1 - R_4} \\ \xrightarrow{R_2 + \frac{2}{3} \times R_4} \\ \xrightarrow{R_3 + R_4} \\ \xrightarrow{R_5 - \frac{5}{3} \times R_4} \\ \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 4500 \\ 0 & 1 & 0 & 0 & 2700 \\ 0 & 0 & 1 & 0 & 3200 \\ 0 & 0 & 0 & 1 & 3600 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, we get the solution $r = 4500$, $b = 2700$, $p = 3200$ and $o = 3600$, which is unique for this system since the bottom row is all 0 elements (including last column). Hence, there are 4500 Red Team fans, 2700 Blue Team fans, 3200 Pink Team fans, and 3600 Orange Team fans.