Grade 7/8 Math Circles
October 27, 2021
Linear and Quadratic Sequences

Definition and Notation

A **sequence** is an ordered list of numbers. For example, the sequence of integers from 1 to 10 is written as $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Each number in the sequence is called a **term**. We use curly brackets to denote a sequence and use commas to separate each term.

A sequence is either *finite* or *infinite*. A *finite* sequence will eventually terminate at a final term (such as the one above). An *infinite* sequence goes on forever. For example,

$$\{1, 3, 5, 7, 9, ...\}$$

is the sequence of all positive odd numbers, and it goes on forever. We use an ellipsis (...) to show that we are omitting information, since it is impossible to write down every term in an infinite sequence.

To refer to a particular term of the sequence, we use t_n, with the subscript n representing the n^{th} term in the sequence. For example, t_1 represents the first term in a sequence.

Exercise 1

Identify t_2, t_5, and t_8 in the sequence below:

$$\{4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34, ...\}$$

Stop and Think

Do some research on the definition of a set. What is the difference between a set and a sequence?

Linear Sequences

You may have noticed a pattern in the sequence introduced in Exercise 1: to get to the next term, simply add 3 to the previous term. In other words, if you take any term in the sequence and subtract the previous term, you obtain the value 3. This value is called the **first difference**.

A **linear sequence** (also known as an **arithmetic sequence**) is a sequence with a *common first difference*, meaning that if you take any term in the sequence and subtract the previous term from it, you will always obtain the same value (the common first difference). The common first difference can be positive, negative, or 0.

Exercise 2

Identify the linear sequences from the options below. For any sequence that is linear, also state the first differences.

(a) $\{-5, 0, 5, 10\}$

(b) $\left\{\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\right\}$

(c) $\{1, 5, 1, 5, 1, 5, \ldots\}$

(d) $\{2, 4, 8, 16, 32, 64\}$

(e) $\{9, 9, 9, 9, 9, 9, \ldots\}$

Closed-Form Formula of a Linear Sequence

A **closed-form formula** allows us to calculate the value of a term given the term number n. The closed-form formula of a linear sequence can be written in the form

$$t_n = an + b$$

with a and b being any number and n being the n^{th} term of the sequence.

Note: In order to specify a finite sequence, the formula must include restrictions on n, such as $n \leq 10$. If there are no restrictions on n, we automatically assume that the sequence is infinite.

Suppose the formula of a linear sequence is $t_n = 3n + 1$. Then the first three terms of the sequence are:

- $t_1 = 3(1) + 1 = 4$
- $t_2 = 3(2) + 1 = 7$
- $t_3 = 3(3) + 1 = 10$

Notice that the common first difference in this sequence is 3, which is equal to a.

There is another way to express the formula for terms in a linear sequence which makes use of the first term of the sequence, t_1. Notice that we can express all linear sequences as

$$\{t_1, t_1 + a, t_1 + 2a, t_2 + 3a, ...\}$$

with a being the common first difference. Thus, another closed-form formula for a linear sequence is

$$t_n = t_1 + a(n - 1)$$

Expressing the formula in this form is useful for seeing the starting point of the sequence.

If we expand this formula, we get that

$$\begin{aligned} t_n &= t_1 + an - a \\ &= an + (t_1 - a) \end{aligned}$$

which means that the value of b in the first form is equal to $t_1 - a$ in the second form.

Exercise 3

Express $t_n = 3n + 1$ in $t_n = t_1 + a(n - 1)$ form.

Exercise 4

What is the 18^{th} term of the sequence with the formula $t_n = -2n + 10$?

Finding a Closed-Form Formula for Linear Sequences

Given a linear sequence, it is useful to derive a closed-form formula for it, so that we can compute the n^{th} term without having to compute all the terms that come before it. In this section, we will go over how to find the formula in both $t_n = an + b$ form and $t_n = t_1 + a(n - 1)$ form. Suppose we want to find the general term of the linear sequence

$$\{10, 7, 4, 1, -2, -5, ...\}$$

For the first form, we need to find the value of a and b.

Note: Here, we can see that the common first difference is -3, which means that $a = -3$. For linear sequences, you can simply find a using $a = t_2 - t_1$ and substitute the value of a into one equation to obtain the value of b, but we will go through the entire procedure here to practice using linear systems.

Step 1: Substitute any two terms along with their term number into the formula $t_n = an + b$ to obtain a system of 2 equations and 2 unknown variables (recall the Systems of Equations lesson.). Here, we will use the first two terms along with $n = 1$ and $n = 2$.

Since $t_1 = 10$ and $t_2 = 7$, we obtain the system below.

$$10 = a(1) + b \tag{1}$$
$$7 = a(2) + b \tag{2}$$

If we rewrite the system slightly, we get

$$10 = a + b \tag{1}$$
$$7 = 2a + b \tag{2}$$

Step 2: Solve the system of equations to obtain the value of a and b.

We will use substitution in this example, but you can use any method you like. We can rewrite equation 1 to be $a = 10 - b$ and substitute it into equation 2. Then equation 2 becomes

$$7 = 2(10 - b) + b \tag{3}$$

Note that $a(b + c)$ can be expanded to $a \times b + a \times c$. Expanding and simplifying equation 3 gives us

$$\begin{aligned} 7 &= 2 \times 10 + 2 \times (-b) + b \\ &= 20 - 2b + b \\ &= 20 - b \end{aligned}$$

Therefore,

$$\begin{aligned} b &= 20 - 7 \\ &= 13 \end{aligned}$$

Finally, we can substitute $b = 13$ into either equation (1) or (2) to find the value of a. Let us use equation (1).

$$\begin{aligned} 10 &= a + b \qquad (1) \\ &= a + 13 \end{aligned}$$

Rearranging and solving for a we get

$$a = 10 - 13 = -3$$

as we expected. Therefore, the closed-form formula for the sequence $\{10,\ 7,\ 4,\ 1,\ -2,\ -5,\ ...\}$ in $t_n = an + b$ form is

$$t_n = -3n + 13$$

Now that we have the formula in the first form, it is easy to obtain the second form using $t_1 = 10$ and $a = -3$. So the formula in $t_n = t_1 + a(n-1)$ form is

$$t_n = 10 - 3(n - 1)$$

Exercise 5

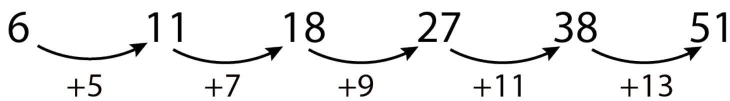
Find the closed-form formula for the sequence $\{-2,\ 5,\ 12,\ 19,\ 26,\ ...\}$ in both $t_n = an + b$ and $t_n = t_1 + a(n-1)$ form.

Quadratic Sequences

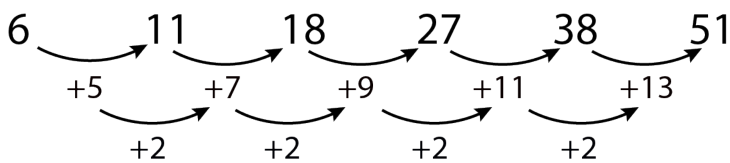
Take a look at the sequence below.

$$\{6, 11, 18, 27, 38, 51, ...\}$$

Let us compute the first differences of this sequence.



Notice that the first differences are not the same. In fact, they are increasing by 2. We can now calculate the **second difference**, which is the difference between two consecutive first differences.



Now we can see that this sequence has a common second difference! Sequences with this characteristic are called quadratic sequences.

A **quadratic sequence** is a sequence with a common, **non-zero** second difference. The common second difference can be positive or negative.

The closed-form formula for a quadratic sequence will always be written in the form

$$t_n = an^2 + bn + c$$

with a, b, and c being any number and n being the term number.

The term n^2 means "n multiplied by itself 2 times", or $n \times n$. This is the term that makes the expression quadratic rather than linear.

Finding a Closed-Form Formula for Quadratic Sequences

Given a quadratic sequence, the process of finding a closed-form formula for it is relatively similar to the method for linear sequences. The major difference is that we will need a system of 3 equations and 3 unknown variables: a, b, and c. We will demonstrate this by finding the formula for the sequence $\{6, 11, 18, 27, 38, 51, ...\}$.

Step 1: Substitute any 3 terms along with their term number into the formula $t_n = an^2 + bn + c$ to obtain a system of 3 equations and 3 unknown variables. Here, we will use the first three terms along with $n = 1$, $n = 2$, and $n = 3$.

Since $t_1 = 6$ and $t_2 = 11$, and $t_3 = 18$ we obtain the system below.

$$\begin{aligned} 6 &= a(1^2) + b(1) + c \qquad (1) \\ 11 &= a(2^2) + b(2) + c \qquad (2) \\ 18 &= a(3^2) + b(3) + c \qquad (3) \end{aligned}$$

Rewriting the system we get

$$\begin{aligned} 6 &= a + b + c \qquad (1) \\ 11 &= 4a + 2b + c \qquad (2) \\ 18 &= 9a + 3b + c \qquad (3) \end{aligned}$$

Step 2: Solve the system of equations to obtain the value of a, b, and c.

Since we used substitution above, let us use matrices this time. Recall from the previous lesson that we can use an augmented matrix to solve systems of equations.

After converting the system into an augmented matrix, we have:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 4 & 2 & 1 & 11 \\ 9 & 3 & 1 & 18 \end{array}\right]$$

Now we can reduce the matrix into RREF:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 4 & 2 & 1 & 11 \\ 9 & 3 & 1 & 18 \end{array}\right] \begin{array}{c} \rightarrow \\ \xrightarrow{R_2 - 4\times R_1} \\ \xrightarrow{R_3 - 9\times R_1} \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -2 & -3 & -13 \\ 0 & -6 & -8 & -36 \end{array}\right] \begin{array}{c} \rightarrow \\ \xrightarrow{R_2 \times -\frac{1}{2}} \\ \rightarrow \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & -6 & -8 & -36 \end{array}\right] \begin{array}{c} \xrightarrow{R_1 - R_2} \\ \rightarrow \\ \xrightarrow{R_3 + 6\times R_2} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} & \frac{13}{2} \\ 0 & 0 & 1 & -3 \end{array}\right] \begin{array}{c} \xrightarrow{R_1 + \frac{1}{2}\times R_3} \\ \xrightarrow{R_2 - \frac{3}{2}\times R_3} \\ \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Using matrices, we have successfully solved the system! We have that $a = 1$, $b = 2$, and $c = 3$.

Therefore, the closed-form formula for our sequence is:

$$t_n = n^2 + 2n + 3$$

Notice that $a = 1$ is half of the second common difference. This is true for all quadratic sequences.

Exercise 6
What is the 20^{th} term in the sequence above? You may use the formula we derived.